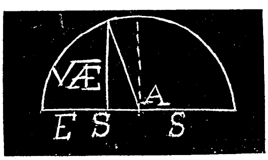
# HISTORICAL NOTE ON THE GRAPHIC REPRESENTATION OF IMAGINARIES BEFORE THE TIME OF WESSEL.

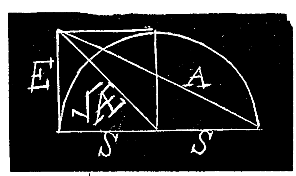
By FLORIAN CAJORI, Colorado College.

John Wallis's attempts at graphic representation of imaginaries have been described by H. Hankel,* W. W. Beman,† D. E. Smith,‡ and more fully by G. Eneström.§ They refer to Wallis's *Treatise of Algebra*, London, 1685, pp. 264-273, but none of them mention Wallis's earlier discussion of this subject in a letter to Collins, May 6, 1673, where he suggests a construction a little different from any of the constructions found in his *Algebra*. This letter was written three or four years before the manuscript of his *Algebra* was ready for print. Wallis says in the letter:||

"This imaginable root in a quadratic equation I have had thoughts long since of designing geometrically, and have had several projects to that purpose. One of them was this: Supposing a quadratic equation $2SA-A^2=Æ$, or (which is equivalent) $A^2-2SA+Æ=0$. If $S\ (=\frac{A+E}{2})$ be bigger than $\surd(Æ)$; that is $S^2>Æ$, the roots are $S\pm\surd(S^2-Æ)=\left\{\begin{matrix}A\\E\end{matrix}\right.$, putting . . . $S=\frac{1}{2}Z=\frac{A+E}{2}$ and . . . $V=\frac{1}{2}X=\frac{A-E}{2}$, where $V\ [=\surd(S^2-Æ)]$, added to and taken from $S$, yields $S+V=A$, $S-V=E$, that is, [the roots are] $S\pm\surd(+V^2)$.




"But if $Æ$ be bigger than $S^2$, the roots are $S\pm\surd(S^2-Æ)\ (=S\pm\surd(-V^2))$, where $\surd(Æ)$, which was the sine, now becomes the secant, and $V$, that was the cosine, is now the tangent. For $S^2\sim Æ=V^2$, the difference of the planes $S^2$ and $Æ$, the greater is to be expressed by the hypotenuse, and the less by the perpendicular."




Evidently, $S$ and Æ are here always positive, hence this is not a general construction of the roots of the quadratic equation.

In both figures the lines $E$ and $A$ represent the roots of the quadratic. In both, the line $E$ extends from the left end of the diameter to the terminal of the line $V$; the line $A$ begins where $E$ ended and extends to the right end of the diameter. Thus the analogy in the construction of real roots in the first figure and of complex roots in the second figure is complete. More-

* Hankel. *Complexe Zahlen*, Leipzig, 1867, pp. 81, 82.

† *Proceedings of the American Association for the Advancement of Science*, Vol. 46, 1897, pp. 35, 36.

‡ Merriman and Woodward, *Higher Mathematics*, 1898, pp. 515, 516.

§ *Bibliotheca Mathematica*, 3rd S., Vol. 7, pp. 263-269.

|| S. J. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century*, Vol. 2, Oxford, 1841, p. 578.

over there is an attempt to secure vector addition. We have $E+A=2S$, also $a+ib+a+ib=2a+2ib$; but we do not have $a+ib+c+id=(a+c)+i(b+d)$. This method of geometric representation of imaginaries labors under another fatal defect, that of representing conjugate roots by lines of different lengths. While this method is less interesting than some others found later in Wallis's *Algebra*, it has one point of superiority over them in being perfectly determinate. Eneström has shown that some of Wallis's geometric notions permit the vector standing for a complex root to take any one of an indefinite number of different directions.* It will be noticed that the present method fails in case of the pure imaginary $\sqrt{(-1)}$, since for that case the radius $S$ of the circle vanishes.

It is well known now that Kühn, in 1750, did not attempt a geometric picture of an imaginary, but simply tried to interpret a negative plane.† He was anticipated by Wallis, who touches upon this point not only in his *Algebra*, but also in the letter of 1673, from which we have been quoting. Wallis says in the letter:

"I was of opinion from the first, that a negative plane may as well be admitted in algebra as a negative length, both being in nature equally impossible; for there can no more be a line less than nothing than a plane less than nothing, both being but imaginable; and if we suppose such a negative square, we may as well suppose it to have a side, not indeed an affirmative, or a negative length, but a supposed mean proportional between a negative and positive thus designable, $\sqrt{(-n)}$, or rather $\sqrt{(-n^2)}$, that is, $\sqrt{(+nx^x-n)}$, a mean proportional between $+n$ and $-n$."

No doubt Wallis touched upon geometric notions on imaginaries in other letters. There is a reference to this subject in one of September 11, 1676.‡ Before this, Collins expressed himself in a letter to J. Gregory, October 19, 1675, as follows:§

"I nothing doubt but the roots of such negative squares, etc., denote an impossibility; as for instance Dr. Wallis, in his first tome, assumes the two legs of a triangle, 2 and 1, to be less than the base 4, and that to show that algebra might be fallacious to a tire, and really finds the segments of the base well; but had he proceeded further he would have found the perpendicular to have been $\sqrt{(-105)}$, which had minifested the impossibility."

A few years ago Felix Müller made the statement that Euler had given a geometric representation of the imaginary by means of a circle.|| Eneström has explained the construction¶ and pointed out that it is the same as one given a hundred years earlier by Wallis, and is in fact trivial. It is quite possible that Euler, like Wallis, may have entertained different

* *Bibliotheca Mathematica*, 3rd S, Vol. 7, p. 266.

† Eneström in *Bibliotheca Mathematica*, 3rd S, Vol. 7, pp. 263-269.

‡ Rigaud, *op. cit.*, Vol. 2, p. 594.

§ Rigaud, *op. cit.*, Vol. 2, p. 278.

|| *Festschr. z. Feier d. 200. Geburtstages L. Eulers*, Leipzig und Berlin, 1907, p. 96; *Opusc. analyt.*, Vol. 2, pp. 76-90.

¶ *Bibliotheca Mathematica*, 3rd S., Vol. 9, 1908-1909, p. 182.

schemes. The query has arisen in the mind of the writer whether not only Euler but also Charles Walmesley had the Gaussian representation in mind when writing the passages we are about to quote. In Euler's great article, *De la controverse entre Leibnitz et Bernoulli*, etc., on the logarithms of complex numbers,* it is explained under "Probleme 4" how to find the anti-logarithm, when the logarithm $g\surd(-1)$ is given. Euler says:

"... pour le trouver on n'a qu'a prendre un arc de cercle$=g$, le rayon etant$=1$ et ayant cherche son sinus et cosinus, le nombre cherche sera $x=\cos g+\surd(-1)\sin g$."

Much the same language is used by Euler in another article in the same volume of the Berlin memoirs (p. 278), *Sur les racines imaginaires des equations*. Six years later Charles Walmesley, a Roman Catholic prelate, wrote an article on logarithms.† To find the logarithm of $a+b\surd(-1)$, he changes it to the form $(a^2+b^2)^{-\frac{1}{2}}(y+u\surd(-1))$, and says, "il est clair qu'on doit prendre l'arc de cercle dont le sinus est $u$, le cosinus $y$," etc.

Did Walmesley and Euler, in connection with $y+ui$, carry in their minds the geometric picture of the lines that were generally used at that time and long before to represent $u$ the sine and $y$ the cosine, the two lines being perpendicular to each other? In other words, did Euler and Walmesley have in mind the Wessel-Argand diagram? There is danger of reading into passages ideas which the authors had not entertained. Hence we must wait for more careful studies of other articles by the same authors, before a final conclusion can be reached.

Another conjecture presents itself. It is asserted that Gauss arrived at the Wessel-Argand diagram independently of Wessel and Argand. We know that Gauss in his doctor's dissertation of 1799 passes in critical review the eighteenth century proofs of the theorem that every equation has a root, including Euler's proof‡ contained in the article *Sur les racines imaginaires des equations*, named above. Our conjecture is that the diagram suggested itself to Gauss upon the reading of the passages in Euler, quoted above.

That the Wessel-Argand diagram was probably in the minds of certain eighteenth century mathematicians before the time of Wessel appears also from a remark of Cauchy,§ to the effect that "a modest scholar," Henri Dominique Truel, as early as the year 1786 represented imaginary quantities upon a line perpendicular to the line for real quantities. We are not aware that Truel ever published his results.

The most interesting and valuable graphic representation of imaginaries before Wessel has never been described by historians of mathematics. It is accomplished by means of a circle and an equilateral hyperbola. Wallis in his *Algebra* suggests various constructions of imaginaries. One of them represents $\surd(1-x^2)$ as the ordinate of the circle $x^2+y^2=1$ when $x<1$, and

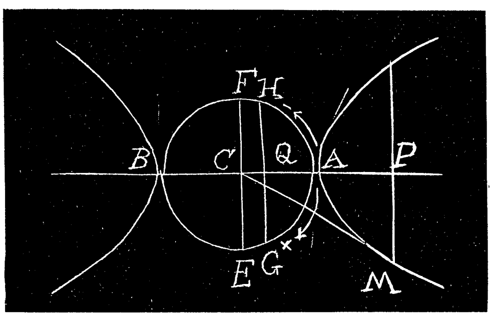
* *Histoire de l'academie r. d. sciences et b. l.*, année 1749, Berlin, 1851, pp. 139-179.

† *Histoire d. l'academie r. d. sciences et b. l.*, année 1755, Berlin, 1757, p. 397.

‡ Ostwald's *Klassiker der Exakt. Wiss.*, No. 14, Leipzig, 1890, pp. 13-21.

§ Cauchy, *Exercises d'analyse et de phys. math.*, T. iv, 1847, p. 157.

as the ordinate of the hyperbola $x^2-y^2=1$ when $x>1$. During the eighteenth century one encounters not infrequently statements to the effect that real (imaginary) arcs of the circle are imaginary (real) arcs of the hyperbola.* These ideas led W. J. G. Karsten in 1768† to the invention of a diagram displaying the infinitely many logarithms of a real or a complex number. He remarks that all ordinates of the circle $x^2+z^2=1$ are imaginary ordinates of the hyperbola $x^2-y^2=1$, where $y=\sqrt{(-1)}z$, that therefore the circle may be regarded as an imaginary part of the hyperbola, and *vice versa*. Consider each of the four arcs coming together at $A$, or at $B$, as the continuation of each of the other three arcs. An arc $AG$ for the circle may be considered as an imaginary arc of the hyperbola. Between any two points $M$ and $G$ exist numberless different arcs, $MAG$, $MAGEBFG$, or in general, $MAG+2\lambda\pi$, and also $MAHBG+2\lambda\pi$, where $\lambda=0, 1, 2, \ldots$ This is true even if $M$, or $G$, or both, coincide with $A$. To any abscissa $x$ there belong therefore not only numberless arcs, but also numberless corresponding sectors. By the Calculus, double the area of a hyperbolic sector corresponding to the abscissa $x$ is$=\log(x+y)=\sqrt{(-1)}.\text{arc}.\cos x$, the sector being assumed zero when $x=1$. If $x>1$, say $x=CP$, then $\log(x+y)$ gives double the sectorial areas corresponding to the arcs $AM\pm\lambda\,(AGBFA)$, that is, $AM\pm2\lambda\pi\sqrt{(-1)}$. Only the sector $ACM$ is real. If $x<1$, then $x+y$ is imaginary. Let $x=CQ$, then $\log(x+y)$ equals double the sectorial areas whose arcs are $AG\pm2\lambda\pi\sqrt{(-1)}$, or $AFBG\pm2\lambda\pi\sqrt{(-1)}$. All of these arcs and sectors are imaginary. From this we see that $\log(-1)$ is represented by double the areas of the sectors belonging to arcs $AEB+\lambda\,(BFAEB)$, or also $AFB+\lambda\,(BEAFB)$.




This graph is in accordance with Euler's formula $\log(-1)=(2\lambda+1)\pi\sqrt{(-1)}$. When $x=0$ and $\lambda=0$, we get the particular logarithm of $\log\sqrt{(-1)}$ which is represented by twice $ACE$. Here then Karsten gives a graphic representation of the well-known Bernoulli-Euler expression $\log\sqrt{(-1)}=\frac{\pi}{2}\sqrt{(-1)}$. In 1842 DeMorgan remarked that "not many years since, [it] was one of the mysteries of analysis," and he apparently believed that he himself had given the earliest geometric interpretation of it.‡ Karsten's diagram gives a geometric picture of the natural logarithms of any real or imaginary number. He also considers the possibility of a graphic representation by using, in place of sec-

* See for instance, Vincenzo Riccati *Sopra logaritmi dei numeri negativi lettere cinque*, Modena, 1779, p. 66. John Playfair in *Phil. Trans.*, Vol. 68, year 1778, Pt. i, pp. 318-343.

† "Abhandlung von den Logarithmen verneinter Grössen," 1. und 2. Abtheil., *Abh. Münch. Akad.* V, 1768

‡ *Transactions of the Cambridge Philosophical Society*, Vol. 7, 1842, p. 294.

tors, the hyperbolic trapezoids (the parallel sides being parallel to one of the asymptotes, the other two sides being the hyperbolic arc and the other asymptote), but he finds this less general and less convenient. Nowhere, either in eighteenth century or nineteenth century authors have we been able to find a reference to Karsten's geometric construction of imaginary logarithms.

## DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

**374. Proposed by H. PRIME, Boston, Massachusetts.**

Divide an angle of 30° into two parts so that the product of the third and fourth powers of their sines (or cosines) shall be a maximum. To be solved without using the methods of calculus. [From *The Maine Farmers' Almanac*, 1912.]

**Solution by H. E. TREFETHEN, Colby College.**

Let $x$ and $30°-x$ be the two parts. $\sin^4 x \sin^3(30°-x)=$maximum, when $\sin^{\frac{4}{3}} x \sin(30°-x)=$maximum, or when $\sin^{\frac{4}{3}} x \cos x-\sin^{\frac{7}{3}} x \sqrt{3}=$maximum. Put $\sin^{\frac{1}{3}} x=y$, $\cos x=(1-y^6)^{\frac{1}{2}}$; and then $y^4(1-y^6)^{\frac{1}{2}} - y^7\sqrt{3}=m$, whence

$$y^{14}-y^8/4+my^7\sqrt{3}/2+m^2/4=0\ldots(1).$$

Let $c$ be a value of $y$ that renders $m$ a maximum. Then the first member of (1) must be exactly divisible *twice* by $y-c$, since a maximum or minimum corresponds to two equal roots. The quotient is readily written down by the synthetic process. The first remainder$=c^{14}-c^8/4+mc^7\sqrt{3}/2+m^2/4=0$, since the division must be exact. Also the second remainder$=14c^{13}-2c^7+7mc^6\sqrt{3}/2=0$.

Eliminating $m$ from the last two equations we get $196c^{14}-203c^8+16c^2=0$. Whence $c^2=0$ corresponding to a minimum, and $196c^{12}-203c^6+16=0$, $c^6=y^6=\sin^2 x=4.813227/56$, or $53.186773/56$. The former gives $x=17°\ 2'\ 52.9''$; the latter ($=\cos^2 x$) gives $x=12°\ 57'\ 7.1''$, and $30°-x=12°\ 57'\ 7.1''$ or $17°\ 2'\ 52.9''$.

It is to be noted that this method for determining maxima and minima can be applied to *any algebraic* expression to which the methods of calculus are applicable.

Also solved by J. Scheffer and A. H. Holmes.