THE SPECIAL FUNCTIONS AND THEIR APPROXIMATIONS

Volume I

This is Volume 53 in
MATHEMATICS IN SCIENCE AND ENGINEERING
A series of monographs and textbooks
Edited by RICHARD BELLMAN, *University of Southern California*

A complete list of the books in this series appears at the end of this volume

THE SPECIAL FUNCTIONS AND THEIR APPROXIMATIONS

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VOLUME I

1969

ACADEMIC PRESS New York and London

ACADEMIC PRESS, INC
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC (LONDON) LTD
Berkeley Square House, London W 1

LIBRARY OF CONGRESS CATALOG CARD NUMBER 68-23498

PRINTED IN THE UNITED STATES OF AMERICA

To My Wife

PREFACE

These volumes are designed to provide scientific workers with a self-contained and unified development for many of the mathematical functions which arise in applied problems, as well as the attendant mathematical theory for their approximations. These functions are often called the special functions of mathematical physics or more simply the special functions.

Although the subject of special functions has a long and varied history, we make no attempt to delve into the many particulars of Bessel functions, Legendre functions, incomplete gamma functions, confluent hypergeometric functions, etc., as these data are available in several sources. We have attempted to give a detailed treatment of the subject on a broad scale on the basis of which many common particulars of the above-named functions, as well as of others, can be derived. Hitherto, much of the material upon which the volumes are based has been available only in papers scattered throughout the literature.

The core of special functions is the Gaussian hypergeometric function ${}_2F_1$ and its confluent forms, the confluent hypergeometric functions ${}_1F_1$ and ψ. The confluent hypergeometric functions slightly modified are also known as Whittaker functions. The ${}_2F_1$ includes as special cases Legendre functions, the incomplete beta function, the complete elliptic functions of the first and second kinds, and most of the classical orthogonal polynomials. The confluent hypergeometric functions include as special cases Bessel functions, parabolic cylinder functions, Coulomb wave functions, and incomplete gamma functions. Numerous properties of confluent hypergeometric functions flow directly from a knowledge of the ${}_2F_1$, and a basic understanding of the ${}_2F_1$ and ${}_1F_1$ is sufficient for the derivation of many characteristics of all the other above-named functions. A natural generalization of the ${}_2F_1$ is the generalized hypergeometric function, the so-called ${}_pF_q$, which in turn is generalized by Meijer's G-function. The theory of the ${}_pF_q$ and the G-function is fundamental in the applications, since they contain as special cases all the commonly used functions of analysis. Further, these functions

are the building blocks for many functions which are not members of the hypergeometric family

The class of hypergeometric series and functions and G-functions considered in these volumes are functions of only a single variable Known generalizations of such hypergeometric series and functions include basic hypergeometric series, hypergeometric series in two or more variables, and G-functions of two or more variables These and other possible generalizations have many important applications, but are not taken up here in view of space requirements Further, the theory of approximations for the above named generalizations analogous to that for functions of a single variable remains to be developed fully

Volume I develops the ${}_2F_1$, ${}_1F_1$, ${}_pF_q$, and the G functions Volume II is mainly concerned with approximations of these functions by series of hypergeometric functions with particular emphasis on expansions in series of Chebyshev polynomials of the first kind, and with the approximations of these functions by the ratio of two polynomials We call the coefficients in the above Chebyshev polynomial expansions "Chebyshev coefficients" Tables of Chebyshev coefficients for numerous special functions are given in Volume II There we also present coefficients which enter into rational approximations for certain special functions

The present work is primarily intended as a reference tool However, much of the material can be used as a text for an advanced undergraduate or graduate course in the special functions and their approximations A two-semester course could be based on the material in Chapters I V and selected topics in Chapters VIII–XI The usual mathematical topics up to and including the residue calculus of complex variable theory are a prerequisite Proofs of many of the key results are given in detail or sketched In a few cases the reader is referred to other sources for proof Often, results are simply stated without proof as they follow essentially from previous results Thus opportunities for exercises are plentiful

In a work of this type, special precautions have been taken to ensure accuracy of all formulas and tables It is a pleasure to acknowledge with thanks the valuable assistance rendered by Mrs Geraldine Coombs and Miss Rosemary Moran in the preparation of the mathematical tables I am particularly grateful to Miss Moran for her assiduous help in proofreading and in preparing the bibliography and indices In spite of all checks imposed to ensure accuracy, it is not reasonable to believe that the text is error-free I would appreciate receiving from readers any criticisms of the material and the identification of any errors

To acknowledge all sources to which some debt is due is virtually

impossible. The bibliography is extensive. For a critical reading of a large portion of the manuscript and numerous suggestions leading to improvement of the text I am indebted to my colleagues Dr. Wyman Fair, Dr. Jerry Fields, and Dr. Jet Wimp. It has been most rewarding to have worked with these same colleagues on many technical papers.

Finally, I am pleased to thank the typist, Mrs. Louise Weston, for her painstaking efforts and devotion to detail in the expert preparation of the manuscript.

YUDELL L. LUKE

Kansas City, Missouri
October, 1968

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INTRODUCTION

To indicate the extent and scope of the present work, and to identify its point of view, a synopsis of the chapters is presented.

Chapter I is devoted to the elements of asymptotic expansions, while Chapter II takes up the gamma function and related functions. The ${}_2F_1$ is studied in Chapter III. There the ${}_pF_q$ is also introduced because many results valid for the ${}_pF_q$ are merely a notational change of results for the ${}_2F_1$. This chapter contains two special features. One is a section on the confluence principle giving conditions so that nontrivial results known for a ${}_pF_q$ can be readily extended to deduce results for an ${}_rF_s$, $r \leqslant p$, $s \leqslant q$. The other feature is the development of Kummer-type relations for the logarithmic solutions of the differential equation satisfied by the ${}_2F_1$, quadratic transformation formulas associated with the logarithmic solutions and evaluation of these solutions for special values of the argument. The features just noted and other relations appear in book form for the first time. Chapter IV studies confluent hypergeometric functions. It is shorter than Chapter III since many results for the confluent functions readily follow from those for the ${}_2F_1$.

The generalized hypergeometric function ${}_pF_q$ and the G-function are the subject of Chapter V. This is a rather long chapter, and by far and large, most of the material has hitherto been available only in research papers. Topics covered include elementary properties, multiplication theorems, integral transforms of the G-function, series of G-functions, expansion theorems, asymptotic expansions of the G-function, and specialization of these results to the asymptotic expansions of the ${}_pF_q$. Results on the G-function are most important since each expression developed becomes a master or key formula from which many results are readily deduced for the more common special functions.

In the applications it often happens that one might know the name of a special function, for example, Struve's function (we call this a "named function"), and would like to know of its properties. It is, therefore, important to identify Struve's function as a ${}_1F_2$. More generally, it is convenient to have an index so that a named function can be identified as a ${}_pF_q$ or as a G-function. On the other hand, [illegible]

a ${}_pF_q$ or a G-function, we would find it helpful to know whether it is one of the well known named special functions To assist the applied worker, we have compiled a list of formulas which serve to identify the ${}_pF_q$ and G-function notation with the named special functions These are presented in Chapter VI There we also give without proof some key properties of Bessel functions, Lommel functions, and the incomplete gamma function and related functions

Asymptotic expansions of the ${}_pF_q$ for large parameters is the subject of Chapter VII The material selected for this chapter is taken from various research papers and is largely governed by results needed in the development of the approximations studied in Volume II

Key properties of the classical orthogonal polynomials are set forth in Chapter VIII These are given without proof, since almost all the results are special cases of data given for the ${}_2F_1$ in Chapter III Topics pertinent to the approximation of functions are presented Special emphasis is given to the evaluation and estimation of coefficients in the expansion of a given function in series of Chebyshev polynomials of the first kind Minimax approximations (that is, best approximations in the Chebyshev sense) are considered and compared with the corresponding truncated expansion in series of Chebyshev polynomials of the first kind The latter are best in the mean square sense Differential and integral characteristics of such expansions are enumerated A nesting procedure is developed to evaluate expeditiously a series of functions where the functions satisfy a linear finite difference equation Thus, expansions in series of orthogonal polynomials can be evaluated in a manner closely akin to the technique used to sum an ordinary polynomial The differential and integral properties of expansions in series of Chebyshev polynomials of the first kind together with the nesting procedure for their evaluation is most important for the applications, since one can operate with such expansions directly as one does with ordinary polynomials without first converting such expansions to an ordinary polynomial

The first eight chapters constitute Volume I In Volume II, expansions of generalized hypergeometric functions and G-functions in series of functions of the same kind is the subject of Chapter IX As special cases we delineate expansions of all the common special functions previously noted in series of Chebyshev polynomials of the first kind These results form the basis for the development of the numerical values of Chebyshev coefficients which are given in Chapter XVII Expansions for many of the special functions in series of Bessel functions are also listed in Chapter IX

Study of rational approximations begins in Chapter X There the

τ-method is introduced and used to get polynomial and rational approximations for the exponential function. For certain values of free parameters, it is shown that the rational approximations coalesce with the approximations which lie on the main diagonal of the Padé table. Padé approximations to the solution of the first-order Riccati equation and to the solution of a generalized second-order Riccati equation are developed. The results for the exponential function are generalized in Chapter XI to get polynomial and rational approximations for the ${}_pF_q$ and for a certain class of G-functions. When $p = 2$, $q = 1$, and one of the numerator parameters is unity, by a special choice of free parameters we recover well-known Padé approximations. These approximations which are equivalent to the truncated continued fractions of Gauss are analyzed in Chapter XIII. Padé approximations for the incomplete gamma functions are detailed in Chapter XIV.

When $p \leqslant q$, the ${}_pF_q(z)$ series converges for all z. But when $p = q + 1$, we have convergence only in the unit disk. However, the function for which the ${}_{q+1}F_q(z)$ series representation is valid only in the unit disk is well defined for all z, $|\arg(1 - z)| < \pi$. This analytically continued function is also called ${}_{q+1}F_q(z)$. The polynomial and rational approximations developed for the ${}_pF_q(z)$ converge for all z when $p \leqslant q + 1$, except that if $p = q + 1$, we must have the restriction $|\arg(1 - z)| < \pi$. Thus, the approximations in the $p = q + 1$ case converge in a domain where the ${}_{q+1}F_q(z)$ series deverges. If $p > q + 1$, and the ${}_pF_q(-z)$ series does not terminate, then it diverges for all $z \neq 0$. In this event, the ${}_pF_q(-z)$ series is the asymptotic expansion of a certain G-function. If $p = q + 2$, the approximations converge for $|\arg z| < \pi/2$ (if $p = 0$ and one of the numerator parameters is unity, we have convergence for $|\arg z| < \pi$), and if $p = q + 3$, we have convergence for $z > 0$. The situation for $p \geqslant q + 2$ is not fully understood. Nonetheless, the information available covers a vast number of special functions. We previously remarked that for a special ${}_2F_1$ and its confluent forms, the rational approximations are of the Padé class. Because both the numerator and denominator polynomials of a Padé approximation satisfy the same three-term recurrence formula, it is natural to inquire if our rational approximations for the ${}_pF_q$ enjoy a similar property. The answer is in the affirmative, and this and related topics are taken up in Chapter XII.

Truncated Chebyshev expansions of Chapter IX are best in the mean square sense, but are not best in the Chebyshev or minimax sense. For virtually all functions of interest in the applications, there is little difference. The Chebyshev coefficients for expansions of the ${}_pF_q$ and for a certain class of G-functions are members of the hypergeometric

family and asymptotic estimates of these coefficients are available Thus a priori evaluation of the effectiveness of such approximations is known In contrast the minimax approximations are not known in closed form except for a few elementary transcendents Thus, in general finite algorithms for the desired coefficients are not available and so they must be found by an iteration process Here tabular values of the function being approximated are required A common way of computing certain transcendents is by Taylor series These are in general only efficient near the point about which the expansion is based Nonetheless, these expansions have the very desirable feature that the $(n+1)$th approximation follows from the nth approximation by a simple addition The rational approximations described above have a like characteristic A striking virtue of the Chebyshev coefficients for the ${}_pF_q$ and for a certain class of G-functions is that they obey a recursion formula of finite length (Chapter XII), and further, in virtually all instances this recursion formula when used in the backward direction can produce numerical values of these coefficients in an efficient manner

For numerous transcendents characterized by definite integrals, use of trapezoidal type integration rules provides an efficient scheme for their computation (Chapter XV)

We have already remarked that expansions in series of Chebyshev polynomials of the first kind can be used in as natural a manner as one uses ordinary polynomial expansions The same is essentially true for the rational approximations Our philosophy of approximations is that they should be as widely applicable in nature as possible They should have application not only for evaluation of the functions and computation of zeros of functions, but they also should be useful to get solutions of differential equations, integral equations, and to invert transforms The potential of these approximations is illustrated with a number of examples in Chapter XVI

In Chapter XVII, we present tables of Chebyshev coefficients for many special functions of both hypergeometric and nonhypergeometric type For a number of special functions of hypergeometric type, coefficients in their rational approximations are presented Some other kinds of coefficients are also given The set of Chebyshev coefficients is the most complete ever assembled Many of these as well as virtually all the coefficients in the rational approximations appear here for the first time

Chapter I ASYMPTOTIC EXPANSIONS

1.1. The Order Symbols O and o

Let z and z_0 be points in a region R of the complex plane. We consider two functions $f(z)$ and $g(z)$ with $g(z) \neq 0$.

If there exists a number A independent of z so that $|f(z)/g(z)| \leqslant A$ for all z in R, then we say that

$$f(z) = O(g(z)) \qquad \text{as} \quad z \to z_0 \quad \text{in } R. \tag{1}$$

Often, where there is no confusion, we omit the qualifying statement "$z \to z_0$ in R." The point z_0 may be at infinity so that we may have $|z| \to \infty$ with $\arg z$ suitably restricted. In illustration, with $f(z) = (1 - \cos z)/z$ and $g(z) = z$,

$$(1 - \cos z)/z = O(z) \qquad \text{as} \quad z \to 0.$$

Also

$$z(1 - \sin z) = O(z) \qquad \text{as} \quad z \to \infty, \quad z \text{ real},$$

$$e^{-z} = O(z^b), \quad 0 < |z| < \infty, \quad |\arg z| \leqslant \pi/2 - \epsilon, \quad \epsilon > 0, \quad b \text{ arbitrary},$$

$$e^{-z} = O(z^b) \quad \text{as} \quad |z| \to \infty, \quad |\arg z| \leqslant \pi/2, \quad R(b) \geqslant 0.$$

If $\lim f(z)/g(z) \to 0$ as $z \to z_0$ in R, then we say that

$$f(z) = o(g(z)) \qquad \text{as} \quad z \to z_0 \quad \text{in } R. \tag{2}$$

Thus, for a previous example, we have

$$(1 - \cos z) = o(z) \qquad \text{as} \quad z \to 0.$$

Also

$$e^{-z} = o(z^a) \qquad \text{as} \quad |z| \to \infty, \quad |\arg z| \leqslant \pi/2 - \epsilon, \quad \epsilon > 0, \quad a \text{ arbitrary}.$$

1.2 Definition of an Asymptotic Expansion

It is convenient to first record what it means when a convergent power series represents the function being expanded If

$$F(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad |z| < r, \tag{1}$$

$$S_n(z) = \sum_{k=0}^{n-1} a_k z^k, \tag{2}$$

then the series on the right of (1) represents $F(z)$ for each fixed z, $|z| < r$, in the sense that

$$\lim_{n\to\infty} (F(z) - S_n(z)) = 0 \tag{3}$$

Thus for z fixed, we can approximate $F(z)$ as closely as we desire by $S_n(z)$ with n sufficiently large Using the order symbols, we have

$$F(z) - S_n(z) = O(z^n) \qquad |z| \leqslant r - \epsilon, \quad \epsilon > 0 \tag{4}$$

$$F(z) - S_n(z) = o(z^{n-1}), \qquad z \to 0 \tag{5}$$

Most always, asymptotic series are of descending type We define an asymptotic power series representation of $F(z)$ as $z \to \infty$, z in some region R, and write

$$F(z) \sim \sum_{k=0}^{\infty} a_k z^{-k} \qquad \text{as} \quad z \to \infty \quad \text{in } R, \tag{6}$$

$$S_n(z) = \sum_{k=0}^{n-1} a_k z^{-k}, \tag{7}$$

if for n fixed,

$$\lim z^{n-1} R_n(z) = \lim z^{n-1}(F(z) - S_n(z)) = 0 \qquad \text{as} \quad z \to \infty \quad \text{in } R \tag{8}$$

That is, we can make $|z^{n-1}R_n(z)| < \epsilon$ with ϵ arbitrarily small and z sufficiently large, z in R Thus for each n fixed, we can approximate $F(z)$ as closely as we desire by taking z sufficiently large, z in R Alternative notations are

$$F(z) = S_n(z) + O(|z^{-n}|) \qquad \text{as} \quad z \to \infty \quad \text{in } R, \tag{9}$$

$$F(z) = S_n(z) + o(|z^{1-n}|) \qquad \text{as} \quad z \to \infty \quad \text{in } R \tag{10}$$

In the above $F(z)$ is said to be asymptotic in the Poincaré sense and the a_k's are the Poincaré coefficients. Asymptotic series are usually divergent, though there is no reason to insist upon this in the definition.

For an illustration, partial integrations show that

$$F(z) = ze^z \int_z^\infty t^{-1}e^{-t}\,dt = S_n(z) + R_n(z),$$

$$S_n(z) = \sum_{k=0}^{n-1} (-)^k k!\, z^{-k}, \tag{11}$$

$$R_n(z) = (-)^n n!\, ze^z \int_z^\infty t^{-n-1}e^{-t}\,dt.$$

The integral for $F(z)$ is defined for all z, $z \neq 0$, $|\arg z| \leqslant \pi/2$. By rotating the path of integration, see the remarks following 1.4(4), we have

$$F(z) = ze^z \int_z^{\infty \exp[i(\theta+\arg z)]} t^{-1}e^{-t}\,dt, \qquad -\pi < \theta < \pi, \quad |\theta + \arg z| < \pi/2. \tag{12}$$

We see that $R_n(z) = O(|z^{-n}|)$ for $|z| \to \infty$, uniformly in $\arg z$, $|\arg z| \leqslant 3\pi/2 - \epsilon$, $\epsilon > 0$, and so

$$F(z) \sim \sum_{k=0}^{\infty} (-)^k k!\, z^{-k}, \qquad |z| \to \infty, \quad |\arg z| \leqslant 3\pi/2 - \epsilon, \quad \epsilon > 0. \tag{13}$$

The asymptotic series is divergent. If z is real and positive, then the error committed when using $S_n(z)$ to approximate $F(z)$ does not exceed in magnitude the last term in $S_n(z)$, and is of opposite sign to this last term. The error is also smaller in magnitude than the first term neglected in the series for $F(z)$, and is of the same sign as this first neglected term.

1.3. Elementary Properties of Asymptotic Series

It is easily shown that if $F(z)$ has an asymptotic expansion in R, it is unique. However, whole classes of functions may have the same asymptotic expansion. To illustrate, in the example 1.2(11), $F(z)$ and $F(z) + e^{-z}$ have the same asymptotic expansion if $|\arg z| < \pi/2$. Of course, $F(z)$ may have a different asymptotic expansion in a region other than R.

If as $z \to \infty$ in R,

$$F(z) \sim \sum_{k=0}^{\infty} a_k z^{-k}, \qquad G(z) \sim \sum_{k=0}^{\infty} b_k z^{-k},$$

then as $z \to \infty$ in R,

$$F(z) + G(z) \sim \sum_{k=0}^{\infty} c_k z^{-k}, \qquad F(z)G(z) \sim \sum_{k=0}^{\infty} d_k z^{-k},$$
$$c_k = a_k + b_k, \qquad d_k = \sum_{k=0}^{n} a_k b_{n-k} \tag{1}$$

A similar result holds for division of $F(z)$ by $G(z)$ provided that $b_0 \neq 0$

Asymptotic expansions may be integrated termwise If as $z \to \infty$ in R,

$$F(z) \sim \sum_{k=0}^{\infty} a_k z^{-k}$$

then as $z \to \infty$ in R,

$$\int_z^{\infty} [F(t) - a_0 - a_1 t^{-1}]\, dt \sim \sum_{k=2}^{\infty} a_k z^{-k+1}/(k-1) \tag{2}$$

If $F(z)$ is differentiable and has the asymptotic expansion as in 1 2(6), and if $F'(z)$ has an asymptotic expansion, then as $z \to \infty$ in R,

$$F'(z) \sim -\sum_{k=1}^{\infty} k a_k z^{-k-1} \tag{3}$$

If $F(z)$ is analytic, the assumption that $F'(z)$ has an asymptotic expansion is not necessary

For proof of (1)–(3), and for a detailed treatment of asymptotic expansions and related topics, see Erdelyi (1956) See also Copson (1965), de Bruijn (1958), Erdelyi and Wyman (1963), Evgrafov (1961), Ford (1960), Froman and Froman (1965) Heading (1962), Jeffreys (1962), Lauwerier (1966), Olver (1954), Wasow (1965), Wilcox (1964), Wyman (1964), and references quoted in these sources

1 4 Watson's Lemma

In this volume, we do not in general discuss methods for obtaining asymptotic expansions This is taken up in the references cited However, it is of interest to examine one special procedure which is applicable when a function is defined by a Laplace integral, since this covers many cases which arise in analysis The result is known as Watson's lemma

Lemma 1. *Let $f(t)$ satisfy the following two conditions:*

(a) $$f(t) = \sum_{k=1}^{\infty} a_k t^{(k/r)-1}, \qquad |t| \leqslant c + \delta,$$

where r, c, and δ are positive.

(b) *There exist positive constants M and b independent of t such that*

$$|f(t)| < Me^{bt}, \quad |t| \geqslant c.$$

Then

$$F(z) = \int_0^{\infty} e^{-zt} f(t)\, dt \sim \sum_{k=1}^{\infty} a_k \Gamma(k/r) z^{-k/r},$$
$$|z| \to \infty, \quad |\arg z| \leqslant \pi/2 - \epsilon, \quad \epsilon > 0. \tag{1}$$

PROOF. We see from the hypotheses that given a fixed positive integer N, we can determine a constant C so that

$$\left| f(t) - \sum_{k=1}^{N} a_k t^{(k/r)-1} \right| \leqslant C t^{(N+1-r)/r} e^{bt}$$

holds for $t \geqslant 0$ independent of the above inequalities between t and c. If

$$F(z) = \sum_{k=1}^{N} \int_0^{\infty} e^{-zt} a_k t^{(k/r)-1}\, dt + R_N$$
$$= \sum_{k=1}^{N} a_k \Gamma(k/r) z^{-(k/r)} + R_N,$$

the result follows once we show that $z^{N/r} R_N \to 0$ as $|z| \to \infty$, $|\arg z| \leqslant \pi/2 - \epsilon$. With $z = x + iy$,

$$|R_N| = \left| \int_0^{\infty} e^{-zt} \left\{ f(t) - \sum_{k=1}^{N} a_k t^{(k/r)-1} \right\} dt \right| \leqslant \int_0^{\infty} e^{-xt} C t^{(N-1-r)/r} e^{bt}\, dt$$
$$= \frac{C\Gamma[(N+1)/r]}{(x-b)^{(N-1)/r}}$$

provided that $(x - b) > 0$. The assumption $|\arg z| \leqslant \pi/2 - \epsilon$ implies that $x \geqslant |z| \sin \epsilon$. So $(x - b) > 0$ if $|z| > b \csc \epsilon$, and under these conditions

$$|z^{N/r} R_N| < \frac{C\Gamma[(N+1)/r]|z|^{N/r}}{(|z| \sin \epsilon - b)^{(N-1)/r}} = O(z^{-1/r}),$$

and the lemma is proved.

We now state and prove a result very much akin to Watson's lemma which is in a form convenient for numerous applications

Lemma 2. *Let $g(t)$ satisfy the following two conditions*

(a) *$g(t)$ is analytic in the sector $|\arg t| < \theta$, and*

$$g(t) = \sum_{k=0}^{\infty} a_k t^k, \qquad |t| < r$$

(b) *$g(t) = O(e^{bt})$ uniformly in $\arg t$ for some b as $|t| \to \infty$ in the sector $|\arg t| < \theta$ Then for*

$$F(z) = \int_0^{\infty} e^{-zt} t^{\sigma-1} g(t)\, dt, \qquad R(\sigma) > 0$$

$$F(z) \sim \Gamma(\sigma) z^{-\sigma} \sum_{k=0}^{\infty} a_k(\sigma)_k z^{-k}, \tag{2}$$

$$|z| \to \infty \qquad |\arg(z-b)| \leqslant \pi/2 + \theta - \epsilon, \qquad \epsilon > 0$$

PROOF The proof of this statement for $|\arg(z-b)| < \pi/2$ is immediate from Watson's lemma We now show that the result also holds in the more extended domain as stated Consider the completely closed contour C which starts from the origin O and proceeds along the real axis to $A = (R, 0)$, $R > 0$, and goes from A to B along a circular arc of radius R with center at the origin, angle $BOA = \varphi$, $-\theta < \varphi < \theta$, and then returns to O along the straight line BO Clearly the integral whose integrand is that of $F(z)$ around C must vanish Now along the arc AB,

$$I = \int_{AB} e^{-zt} t^{\sigma-1} g(t)\, dt = iR^{\sigma} \int_0^{\varphi} \exp\{-R|z-b| \exp(i[\omega + \arg(z-b)])\}$$
$$\times e^{i\omega\sigma} \exp\{-Rbe^{i\omega}\} g(Re^{i\omega})\, d\omega,$$

and for $R_0 > R$,

$$|I| \leqslant MR_0^{\sigma} \int_0^{\varphi} \exp\{-R_0 |z-b| \cos[\omega + \arg(z-b)]\}\, d\omega$$

where M is a constant independent of R_0 If $|\varphi + \arg(z-b)| < \pi/2$, $I \to 0$ as $R \to \infty$ Hence, as $-\theta < \varphi < \theta$,

$$F(z) = \int_0^{\infty} e^{-zt} t^{\sigma-1} g(t)\, dt = \int_0^{\infty e^{i\varphi}} e^{-zt} t^{\sigma-1} g(t)\, dt$$
$$= e^{i\varphi\sigma} \int_0^{\infty} \exp\{-zte^{i\varphi}\} t^{\sigma-1} g(te^{i\varphi})\, dt, \tag{3}$$

$$R(\sigma) > 0 \qquad |\arg(z-b)| \leqslant \pi/2 + \theta - \epsilon, \qquad \epsilon > 0$$

This shows that the domain of validity of the Laplace integral can be extended by rotating the path of integration. The hypotheses of Watson's lemma for the last integral in (3) are satisfied, and so

$$F(z) \sim e^{i\varphi\sigma} \sum_{k=0}^{\infty} a_k e^{i\varphi k} \int_0^{\infty} \exp\{-zte^{i\varphi}\} t^{\sigma+k-1}\, dt$$

which leads to (2).

Note that in the example 1.2(11), if we replace t by $z(1 + t)$, then

$$F(z) = z \int_0^{\infty} [e^{-zt}/(1 + t)]\, dt. \tag{4}$$

Now $f(t) = (1 + t)^{-1} = \sum_{k=0}^{\infty} (-)^k t^k$ if $| t | < 1$, and so

$$f(t) = \sum_{k=0}^{\infty} (-)^k t^k \qquad \text{if} \quad | t | \leqslant \tfrac{2}{3}.$$

Thus in the notation of Watson's lemma, $a_k = (-)^k$, $c = \frac{1}{2}$, $\delta = \frac{1}{6}$. Also $r = 1$. If $t \geqslant \frac{1}{2}$, $f(t) \leqslant \frac{2}{3} < \frac{2}{3}e^{bt}$ and in the notation of Watson's lemma, $M = \frac{2}{3}$ and $b > 0$, but arbitrarily small. The hypotheses of Watson's lemma are fulfilled and we get 1.2(11) for $| \arg z | \leqslant \pi/2 - \epsilon$, and by rotating the path of integration, see (3), the asymptotic expansion is valid in the extended domain $| \arg z | \leqslant 3\pi/2 - \epsilon$, $\epsilon > 0$.

Chapter II THE GAMMA FUNCTION AND RELATED FUNCTIONS

2.1. Definitions and Elementary Properties

The gamma function can be defined by Euler's integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad R(z) > 0 \tag{1}$$

or equivalently by the Laplace integral

$$\Gamma(z) = p^z \int_0^\infty e^{-pt} t^{z-1}\, dt, \qquad R(p) > 0, \quad R(z) > 0,$$

$$R(p) = 0 \qquad \text{if} \quad 0 < R(z) < 1 \tag{2}$$

By rotating the path of integration, see 1.4(3), we have

$$\Gamma(z) = p^z \int_0^{\infty e^{i\theta}} e^{-pt} t^{z-1}\, dt \tag{3}$$

$$|\theta + \arg p| < \pi/2, \quad R(z) > 0, \qquad |\theta + \arg p| = \pi/2 \quad \text{if} \quad 0 < R(z) < 1$$

Partial integration of (1) shows that $\Gamma(z)$ satisfies the difference equation

$$\Gamma(z+1) = z\Gamma(z) \tag{4}$$

and since $\Gamma(1) = \Gamma(2) = 1$,

$$\Gamma(n+1) = 1 \cdot 2 \cdots n = n! \tag{5}$$

It is convenient to introduce the notation

$$(a)_k = a(a+1) \cdots (a+k-1), \qquad (a)_0 = 1 \tag{6}$$

Then the formulas

$$(a)_k = \Gamma(a+k)/\Gamma(a), \tag{7}$$

$$(a)_{n+k} = (a+n)_k(a)_n, \tag{8}$$

$$(a)_{n-k} = \frac{(-)^{n+k}}{(1-a)_{-n}(1-a-n)_k} = \frac{(-)^k(a)_n}{(1-a-n)_k}, \tag{9}$$

$$(a)_{nk} = k^{nk}\left(\frac{a}{k}\right)_n\left(\frac{a+1}{k}\right)_n \cdots \left(\frac{a+k-1}{k}\right)_n, \tag{10}$$

are easily proved for integer values of n and k. However, (7)–(9) have meaning for general values of n and k so long as the gamma functions involved are defined. The binomial coefficient is defined by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{(-)^k(-n)_k}{k!}. \tag{11}$$

$\Gamma(z)$ can also be defined by the formulas

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\cdots(z+n)} = \lim_{n\to\infty} \frac{n^z}{z(1+z)(1+\frac{1}{2}z)\cdots(1+z/n)} \\ &= \lim_{n\to\infty} \frac{[(1+1)(1+\frac{1}{2})\cdots(1+(1/n-1))]^z}{z(1+z/n)[(1+z)(1+\frac{1}{2}z)\cdots(1+(z/n-1))]} \\ &= z^{-1}\prod_{n=1}^{\infty}(1+1/n)^z(1+z/n)^{-1},\end{aligned} \tag{12}$$

$$1/\Gamma(z) = ze^{\gamma z}\prod_{n=1}^{\infty}[(1+z/n)e^{-z/n}], \tag{13}$$

where γ is the Euler–Mascheroni constant, and

$$\gamma = \lim_{m\to\infty} a_m, \qquad a_m = \sum_{k=1}^{m} k^{-1} - \ln m. \tag{14}$$

The a_m's form a decreasing sequence since

$$a_{m+1} - a_m = \frac{1}{m+1} + \ln\left(1 - \frac{1}{m+1}\right) = -\sum_{n=1}^{\infty}[(n+1)(m+1)^{n+1}]^{-1} < 0.$$

If $m \geqslant 2$, $m^{-1} < \int_{m-1}^{m} t^{-1}\,dt < (m-1)^{-1}$ and by summation of these inequalities, it follows that $m^{-1} < a_m < 1$. Thus γ exists and $0 < \gamma < 1$. We will prove the equivalence of (1), (12), and (13), but first we note

from the latter that $\Gamma(z)$ is analytic everywhere in the bounded complex plane save for z a negative integer or zero, at which points $\Gamma(z)$ has simple poles The point $z = \infty$ is an essential singularity of $\Gamma(z)$ From (13),

$$1/\Gamma(z) = z \lim_{m\to\infty} \left[\exp\{(1 + \tfrac{1}{2} + \quad + 1/m - \ln m)z\} \prod_{n=1}^{m} (1 + z/n)e^{-z/n}\right]$$

$$= z \lim_{m\to\infty} \left[m^{-z} \prod_{n=1}^{m} (1 + z/m)\right]$$

and this is the reciprocal of the first equality in (12) Thus (12) and (13) are the same If n is a positive integer and $R(z) > 0$, then repeated integration by parts gives

$$\int_0^n (1 - t/n)^n t^{z-1}\, dt = b_n, \qquad b_n = \frac{n!\, n^z}{z(z+1) \quad (z+n)}$$

Now

$$\Gamma(z) - b_n = \int_0^n \{e^{-t} - (1 - t/n)^n\} t^{z-1}\, dt + \int_n^\infty e^{-t} t^{z-1}\, dt,$$

and the second integral approaches 0 as $n \to \infty$ The same is true for the first integral, see Whittaker and Watson (1927, p 242) or Rainville (1960, pp 15–18) Thus the equivalence of (1) and (12) follows

In place of (12) we can also write

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{z(z+1) \quad (z+n-1)} = \lim_{n\to\infty} \frac{(n-1)!\, n^z}{(z)_n} \tag{15}$$

in view of (7), and this implies that

$$\lim_{n\to\infty} \frac{(n-1)!\, n^z}{\Gamma(z+n)} = 1 \tag{16}$$

2.2. Analytic Continuation of $\Gamma(z)$

We now show how values of $\Gamma(z)$ for z in the left half-plane can be deduced from values of $\Gamma(z)$ for z in the right half-plane From 2 1(12) and a known result on an infinite product [see Copson (1955, p 150)],

$$[\Gamma(z)\Gamma(-z)]^{-1} = -z^2 \prod_{n=1}^{\infty} (1 - z^2/n^2) = -(z/\pi) \sin \pi z,$$

whence

$$\Gamma(z)\Gamma(-z) = -(\pi/z) \csc \pi z \tag{1}$$

Thus also,

$$\Gamma(z)\Gamma(1-z) = \pi \csc \pi z, \tag{2}$$

$$\Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z) = \pi \sec \pi z, \tag{3}$$

$$\Gamma(\tfrac{1}{2}) = \pi^{1/2}. \tag{4}$$

Further,

$$|\Gamma(n+1+iy)| = \{(\pi y/\sinh \pi y)(1^2+y^2)(2^2+y^2)\cdots(n^2+y^2)\}^{1/2}, \tag{5}$$

$$|\Gamma(n+\tfrac{1}{2}+iy)| = \{(\pi/\cosh \pi y)(\tfrac{1}{4}+y^2)(\tfrac{9}{4}+y^2)\cdots[(n-\tfrac{1}{2})^2+y^2]\}^{1/2}. \tag{6}$$

2.3. Multiplication Formula

We now prove that

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)} m^{mz-\frac{1}{2}} \prod_{r=0}^{m-1} \Gamma(z+r/m), \tag{1}$$

which for $m = 2$ is the duplication formula

$$\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2})/\pi^{\frac{1}{2}} = 2^{2z-1}(1)_z(\tfrac{1}{2})_z\,. \tag{2}$$

For the proof, let

$$A(z) = \frac{m^{mz}\prod_{r=0}^{m-1}\Gamma(z+r/m)}{m\Gamma(mz)},$$

and combine this with 2.1(15) to write

$$A(z) = \frac{m^{mz}\prod_{r=0}^{m-1}\lim_{n\to\infty}\dfrac{(n-1)!\,n^{z+r/m}}{(z+r/m)_n}}{m\lim_{n\to\infty}\dfrac{(nm-1)!\,(nm)^{mz}}{(mz)_{nm}}}.$$

Now use 2.1(9) with $a = mz$ and $k = m$. Thus,

$$A(z) = \lim_{n\to\infty}\frac{m^{nm-1}[(n-1)!]^m n^{(m-1)/2}}{(nm-1)!}$$

which is independent of z and so a constant To evaluate the constant, put $z = 1/m$ Thus,

$$A(1/m) = \prod_{r=0}^{m-1} \Gamma[(1+r)/m] = \prod_{r=0}^{m-1} \Gamma[(m-r)/m],$$

$$[A(1/m)]^2 = \prod_{r=1}^{m-1} \Gamma(r/m)\Gamma(1-r/m) = \pi^{m-1} \Big/ \prod_{r=1}^{m-1} \sin(\pi r/m)$$

in view of 2 2(2) Now

$$x^{2m} - 2x^m \cos m\theta + 1 = \prod_{r=1}^{m} \{x^2 - 2x\cos(\theta + 2(r-1)\pi/m) + 1\}$$

since $x = \exp[\pm i(\theta + 2(r-1)\pi/m)]$, $r = 1, 2, \ldots, m$ are the $(2m)$ zeros of the left-hand side of the latter equation Let $x = 1$ We find that

$$\prod_{r=1}^{m-1} \sin\left(\frac{\theta}{2} + \frac{r\pi}{m}\right) = \frac{m}{2^{m-1}}\left(\frac{\sin m\theta/2}{m\theta/2}\right)\left(\frac{\theta/2}{\sin\theta/2}\right),$$

and so $A(1/m) = (2\pi)^{(m-1)/2} m^{-1/2}$ and (1) follows

2.4. The Logarithmic Derivative of the Gamma Function

This is notated as $\psi(z)$ Thus,

$$\psi(z) = (d/dz)\ln\Gamma(z) = \Gamma'(z)/\Gamma(z) \qquad \text{or} \quad \ln\Gamma(z) = \int_1^z \psi(t)\,dt \tag{1}$$

Numerous results for $\psi(z)$ follow by differentiation of formulas for $\Gamma(z)$ Thus from 2 1(12–13),

$$\psi(z) = \lim_{n\to\infty}\left[\ln n - \sum_{k=0}^{n} (z+k)^{-1}\right], \tag{2}$$

$$\psi(z) = -\gamma - 1/z + \sum_{k=1}^{\infty} z/k(z+k)$$

$$= -\gamma + (z-1)\sum_{k=0}^{\infty} [(k+1)(z+k)]^{-1} \tag{3}$$

Clearly $\psi(z)$ is analytic everywhere in the bounded complex plane except for simple poles at $z = 0, -1, -2, \ldots$ Also

$$\psi(1) = -\gamma \tag{4}$$

The following are easily proved from results for $\Gamma(z)$; we omit details:

$$\psi(z+n) = 1/z + 1/(z+1) + \cdots + 1/(z+n-1) + \psi(z), \quad n = 1, 2, 3, \ldots . \tag{5}$$

$$\psi(1+n) = 1 + \tfrac{1}{2} + \cdots + (1/n) - \gamma. \tag{6}$$

$$\psi(z) - \psi(-z) = -\pi \cot \pi z - 1/z. \tag{7}$$

$$\psi(z) - \psi(1-z) = -\pi \cot \pi z. \tag{8}$$

$$\psi(\tfrac{1}{2} + z) - \psi(\tfrac{1}{2} - z) = \pi \tan \pi z. \tag{9}$$

$$\psi(mz) = m^{-1} \sum_{k=0}^{m-1} \psi(z + k/m) + \ln m. \tag{10}$$

With $z = \frac{1}{2}$ and $m = 2$, the latter gives

$$\psi(\tfrac{1}{2}) = -\gamma - 2 \ln 2. \tag{11}$$

This is a special case of the formula

$$\psi(p/q) = -\gamma - \ln q - \tfrac{1}{2}\pi \cot(\pi p/q) + \sum_{n=1}^{[\frac{1}{2}q]}{}' \cos(2\pi pn/q) \ln[2 - 2\cos 2\pi n/q], \tag{12}$$

where p and q are positive integers, $0 < p < q$. The prime attached to the summation index indicates that if q is even, the last term in the sum is taken with half weight. For proof, see Erdélyi *et al.* (1953, Vol. 1, p. 19), or Nielsen (1965, Vol. 1, p. 20).

If m and k are positive integers or zero,

$$\lim_{z \to -m} [\psi(z)/\Gamma(z-k)] = (-)^{m+k+1}(m+k)!, \tag{13}$$

$$\begin{aligned} \lim_{z \to -m} (z)_k(\psi(z+k) - \psi(z)) &= (-m)_k(\psi(1+m-k) - \psi(1+m)) \quad \text{if } k \leqslant m \\ &= (-)^m m!(k-m-1)! \quad \text{if } k > m. \end{aligned} \tag{14}$$

2.5. Integral Representations for $\psi(z)$ and $\ln \Gamma(z)$

Of numerous integral representations for $\psi(z)$, we prove

$$\psi(z) = \int_0^\infty [t^{-1}e^{-t} - (1-e^{-t})^{-1}e^{-tz}]\, dt, \qquad R(z) > 0, \tag{1}$$

$$\psi(z) = \ln z + \int_0^\infty [t^{-1} - (1-e^{-t})^{-1}]e^{-tz}\, dt, \qquad R(z) > 0, \tag{2}$$

$$\psi(z) = \ln z - \tfrac{1}{2}z^{-1} - \int_0^\infty [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}]e^{-tz}\, dt, \qquad R(z) > 0. \tag{3}$$

The formula

$$\ln n = \int_0^\infty (e^{-t} - e^{-nt})t^{-1}\,dt \tag{4}$$

follows by integration of

$$x^{-1} = \int_0^\infty e^{-xt}\,dt$$

with respect to x from 1 to n Use the latter with $x = z + k$ and (4) in 2 4(2) to get

$$\psi(z) = \lim_{n\to\infty}\left[\int_0^\infty \left\{(e^{-t} - e^{-nt})t^{-1} - \sum_{k=0}^{n} e^{-t(z+k)}\right\} dt\right],$$

$$\psi(z) = \lim_{n\to\infty}\left[\int_0^\infty \{t^{-1}e^{-t} - (1 - e^{-t})e^{-tz}\}\,dt\right.$$

$$\left. - \int_0^\infty e^{-nt}\{t^{-1} - (1 - e^{-t})^{-1}e^{-t(z+1)}\}\,dt\right]$$

Here, the first integral is independent of n, and the second integral approaches zero as $n \to \infty$ This proves (1) Now use (4) with $n = z$ and (1) to obtain (2) Equation (3) is a simple rearrangement of (2)

Next we prove that

$$\ln \Gamma(z) = (z - \tfrac{1}{2})\ln z - z + \tfrac{1}{2}\ln(2\pi) + \int_0^{\infty e^{i\theta}} [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}]t^{-1}e^{-tz}\,dt,$$

$$-\pi/2 < \theta < \pi/2 \qquad -(\pi/2 + \theta) < \arg z < \pi/2 - \theta \tag{5}$$

From 2 4(1) and (3), we get

$$\ln \Gamma(z) = (z - \tfrac{1}{2})\ln z - z + 1 + \int_0^\infty h(t)t^{-1}e^{-tz}\,dt - \int_0^\infty h(t)t^{-1}e^{-t}\,dt,$$

$$h(t) = [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}]$$

Put $z = \frac{1}{2}$ and use 2 2(4) Let J and I be $\int_0^\infty h(t)e^{-tz}\,dt$ with $z = \frac{1}{2}$ and $z = 1$, respectively Then

$$\tfrac{1}{2}(\ln \pi - 1) = J - I = \int_0^\infty h(t)t^{-1}e^{-t/2}\,dt - \int_0^\infty h(t)t^{-1}e^{-t}\,dt$$

$$= \int_0^\infty [h(t) - h(t/2)]t^{-1}e^{-t}\,dt = \int_0^\infty [t^{-1}e^{-t/2} - (e^t - 1)^{-1}]t^{-1}\,dt$$

Hence,

$$J = \int_0^\infty [t^{-1}(e^{-t/2} - e^{-t}) - \tfrac{1}{2}e^{-t}]t^{-1}\,dt$$

$$= -\int_0^\infty (d/dt)[t^{-1}(e^{-t/2} - e^{-t})]\,dt + \tfrac{1}{2}\int_0^\infty [e^{-t} - e^{-t/2}]t^{-1}\,dt$$

Thus in view of (4), $J = (1 - \ln 2)/2$ and so $I = 1 - \frac{1}{2}\ln(2\pi)$ and (5) follows for $\theta = 0$. The more general statement obtains by rotating the path of integration. Another expression for $\ln \Gamma(z)$ is

$$\ln \Gamma(z) = (z - \tfrac{1}{2}) \ln z - z + \tfrac{1}{2} \ln(2\pi) + 2 \int_0^\infty [\arctan (t/z)](e^{2\pi t} - 1)^{-1}\, dt. \quad (6)$$

For proof, see Erdélyi *et al.* (1953, Vol. 1, p. 22).

2.6. The Beta Function and Related Functions

The beta function is defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1 - t)^{\beta-1}\, dt, \qquad R(\alpha) > 0, \quad R(\beta) > 0, \quad (1)$$

or in trigonometric form

$$B(\alpha, \beta) = 2 \int_0^{\pi/2} (\sin \theta)^{2\alpha-1}(\cos \theta)^{2\beta-1}\, d\theta, \qquad R(\alpha) > 0, \quad R(\beta) > 0. \quad (2)$$

Clearly $B(\alpha, \beta) = B(\beta, \alpha)$ and we prove that

$$B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta). \quad (3)$$

From 2.1(1),

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty e^{-t}t^{\alpha-1}\, dt \int_0^\infty e^{-u}u^{\beta-1}\, du.$$

In the integrands, replace t by x^2 and u by y^2, and then transfer to polar coordinates with $x = r \cos \theta$, $y = r \sin \theta$. Thus, we can write

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= 2 \int_0^\infty e^{-r^2}r^{2\alpha+2\beta-1}\, dr\, 2 \int_0^{\pi/2} \cos^{2\alpha-1}\theta \sin^{2\beta-1}\theta\, d\theta \\ &= \left(\int_0^\infty e^{-t}t^{\alpha+\beta-1}\, dt\right)B(\beta, \alpha) = \Gamma(\alpha + \beta)B(\beta, \alpha)\end{aligned}$$

which proves (1).

Consider $\int_C (z^{-1} - z)^\alpha z^{\beta-1}\, dz$ where C is the contour consisting of the semicircle $|z| = 1$ in the upper half-plane and its diameter with indentations of radius ϵ at the points $z = 0, \pm 1$. Let $\epsilon \to 0$ and get

$$\int_0^\pi (\sin t)^\alpha e^{it\beta}\, dt = \frac{\pi e^{\frac{1}{2}i\pi\beta}\Gamma(1 + \alpha)}{2^\alpha \Gamma[1 + \frac{1}{2}(\alpha + \beta)]\Gamma[1 + \frac{1}{2}(\alpha - \beta)]}, \qquad R(\alpha) > -1. \quad (4)$$

Now consider $\int_C (z^{-1} + z)^\alpha z^{\beta-1}\, dz$ where C is the semicircle $|z| = 1$ in the right half-plane and its diameter with indentations of radius ϵ at the points $z = 0, \pm i$ Let $\epsilon \to 0$ and obtain

$$\int_0^{\pi/2} (\cos t)^\alpha \cos \beta t\, dt = \frac{\pi\Gamma(1+\alpha)}{2^{\alpha+1}\Gamma[1+\frac{1}{2}(\alpha+\beta)]\Gamma[1+\frac{1}{2}(\alpha-\beta)]}, \qquad R(\alpha) > -1 \tag{5}$$

2 7. Contour Integral Representations for Gamma and Beta Functions

We use the notation $\int_\sigma^{(0+)} f(t)\, dt$ to designate a loop integral where the path of integration C starts at σ, encircles the origin once in the counterclockwise direction, and returns to σ We suppose that no singularities of $f(t)$ except those at $t = 0$ are within the contour If $\sigma = -R, R > 0$, then C may be taken as a line from $-R$ to $-r$, with $t = ue^{-i\pi}$, the circle $t = re^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$, and the line from $-r$ to $-R$ with $t = ue^{i\pi}$ If $f(t) = e^t t^{-z}$, then

$$\int_{-\infty}^{(0+)} e^t t^{-z}\, dt = -e^{i\pi z}\int_\infty^r e^{-u}u^{-z}\, du - e^{-i\pi z}\int_r^\infty e^{-u}u^{-z}\, du$$
$$- ir^{1-z}\int_{-\pi}^{\pi} \exp[re^{i\theta}]e^{i\theta(1-z)}\, d\theta$$

If $R(z) < 1$, the third integral goes to zero with r, and so with the aid of 2 1(1) and 2 2(2), we have

$$[\Gamma(z)]^{-1} = \pi^{-1}(\sin \pi z)\Gamma(1-z) = (2\pi i)^{-1}\int_{-\infty}^{(0+)} e^t t^{-z}\, dt, \qquad |\arg t| \leqslant \pi \tag{1}$$

Note that each side of the equations in (1) represent entire functions of z Thus the restriction $R(z) < 1$ may be dropped and (1) is valid for all z Replace z by $(1 - z)$ in (1) Then

$$2i(\sin \pi z)\Gamma(z) = \int_{-\infty}^{(0+)} e^t t^{z-1}\, dt \qquad |\arg t| \leqslant \pi \tag{2}$$

Also

$$2i(\sin \pi z)\Gamma(z) = -\int_\infty^{(0+)} e^{-t}(-t)^{z-1}\, dt \qquad |\arg(-t)| \leqslant \pi \tag{3}$$

Now consider $(2\pi i)^{-1}\int_B f(t)\, dt$ where B is a completely closed contour composed of a line from $c - iR$ to $c + iR, c > 0, R > 0$, a line from

$c + iR$ to iR, a quarter circle in the upper left half-plane with radius R and center at the origin, the reverse of the contour C described above with $\sigma = -R$, a quarter circle in the lower left half-plane with radius R and center at the origin, and the line from $-iR$ to $c - iR$.

Again let $f(t) = e^t t^{-z}$. Then

$$\int_{c+iR}^{iR} f(t)\,dt = e^{iR}\int_c^0 e^u(u + iR)^{-z}\,du$$

which goes to zero as $R \to \infty$, if $R(z) > 0$. For the integral along the quarter circle in the upper half-plane with $t = Re^{i\theta}$, we get

$$V = -R^{1-z}e^{-i\pi z/2}\int_0^{\pi/2} \exp\{-R\sin\varphi + iR\cos\varphi + i\varphi(1 - z)\}\,d\varphi.$$

Now $2/\pi \leqslant (\sin\varphi)/\varphi \leqslant 1$ for $0 \leqslant \varphi \leqslant \pi/2$. Hence with M a positive number independent of R,

$$|V| \leqslant M|R^{1-z}|\int_0^{\pi/2} \exp(-R\sin\varphi)\,d\varphi < M|R^{1-z}|\int_0^{\pi/2} \exp(-2R\varphi/\pi)\,d\varphi,$$

$$|V| < \pi M|R^{-z}|(1 - e^{-R}).$$

So if $R(z) > 0$, the integrals taken round the quarter circles vanish when $R \to \infty$. It follows from Cauchy's theorem that

$$(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} e^t t^{-z}\,dt = (2\pi i)^{-1}\int_{-\infty}^{(0+)} e^t t^{-z}\,dt,$$

$$c > 0, \qquad |\arg t| \leqslant \pi, \qquad 0 < R(z) < 1. \tag{4}$$

From (1) and (4),

$$[\Gamma(z)]^{-1} = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} e^t t^{-z}\,dt, \qquad c > 0, \quad R(z) > 0, \tag{5}$$

where that part of the restriction $R(z) < 1$ may be dropped by analytic continuation.

The integrals (1)–(3) may be generalized by rotating the path of integration. Thus consider $\int_{-\infty e^{i\delta}}^{(0+)} t^{z-1}e^{-\sigma t}\,dt$ where the initial and final values of $\arg t$ are $\delta - \pi$ and $\delta + \pi$, respectively. Then

$$\Gamma(z) = \sigma^z(e^{2\pi i z} - 1)^{-1}\int_{-\infty e^{i\delta}}^{(0+)} t^{z-1}e^{-\sigma t}\,dt,$$

$$\pi/2 - \delta < \arg\sigma < 3\pi/2 - \delta, \qquad \delta - \pi \leqslant \arg t \leqslant \delta + \pi, \qquad z \neq 0, \pm 1, \pm 2, \ldots, \tag{6}$$

and upon replacing z by $(1 - z)$, we find that for all values of z,

$$2\pi i(\sigma e^{-i\pi})^{z-1}[\Gamma(z)]^{-1} = \int_{-\infty e^{i\delta}}^{(0+)} t^{-z}e^{-\sigma t}\,dt,$$
$$\pi/2 - \delta < \arg\sigma < 3\pi/2 - \delta, \qquad \delta - \pi \leqslant \arg t \leqslant \delta + \pi \tag{7}$$

Now consider $\int_C f(t)\,dt$, $f(t) = t^{x-1}(1-t)^{y-1}$, where the contour C starts from a point B on the real axis between $t = 0$ and $t = 1$ and consists of a loop around $t = 1$ in the positive direction, a loop around $t = 0$ in the positive direction, a loop around $t = 1$ in the negative direction, and a loop around $t = 0$ in the negative direction, so that $f(t)$ returns to B with its initial value Let the loop around $t = 1$ be the line from B to $1 - \rho$, the circle $|t - 1| = \rho$, and the line from $(1 - \rho)$ to B, and similarly for the other loops Let $\rho \to 0$ Then with the notation $\int_C f(t)\,dt = \int^{(1+\,0+\,1-\,0-)} f(t)\,dt$, we have

$$B(x, y) = -\frac{e^{-i\pi(x+y)}}{4\sin\pi x\sin\pi y}\int^{(1+\,0+\,1-\,0-)} t^{x-1}(1-t)^{y-1}\,dt, \tag{8}$$

provided that neither x nor y is an integer or zero

Similarly, $B(x, y)$ can be represented by single loop integrals Thus,

$$B(x, y) = (2i)^{-1}\csc\pi y\int_0^{(1+)} t^{x-1}(t-1)^{y-1}\,dt,$$
$$|\arg(t-1)| \leqslant \pi, \qquad R(x) > 0, \qquad y \neq 0, \pm 1, \pm 2, \tag{9}$$

$$B(x, y) = -(2i)^{-1}\csc\pi x\int_1^{(0+)} (-t)^{x-1}(1-t)^{y-1}\,dt,$$
$$|\arg(-t)| \leqslant \pi, \qquad R(y) > 0, \qquad x \neq 0, \pm 1, \pm 2, \tag{10}$$

In (9), let $t = e^v$ so that $t = 0$ corresponds to $-\infty e^{i\delta}$ with δ real and $|\delta| < \pi/2$ Then

$$B(x, y) = (2i)^{-1}\csc\pi y\int_{-\infty e^{i\delta}}^{(0+)} e^{vx}(e^v - 1)^{y-1}\,dv,$$
$$-\pi/2 < \delta < \pi/2$$
$$\delta - \pi < \arg(e^v - 1) \leqslant \delta + \pi, \qquad R(xe^{i\delta}) > 0 \qquad y \neq 1, 2, 3, \tag{11}$$

2 8. Bernoulli Polynomials and Numbers

The generalized Bernoulli polynomials $B_k^{(a)}(x)$ can be defined by the generating formula

$$\frac{t^a e^{xt}}{(e^t - 1)^a} = \sum_{k=0}^{\infty}\frac{t^k}{k!}B_k^{(a)}(x), \qquad |t| < 2\pi \tag{1}$$

Since

$$B_k^{(a)}(x) = \frac{d^k}{dt^k}\left\{\left(\frac{t}{e^t-1}\right)^a e^{xt}\right\}_{t=0} = \sum_{j=0}^{k}\binom{k}{j}\left\{\frac{d^{k-j}}{dt^{k-j}}(e^{xt})\frac{d^j}{dt^j}\left(\frac{t}{e^t-1}\right)^a\right\}_{t=0}$$

$$= \sum_{j=0}^{k}\binom{k}{j}x^{k-j}\left\{\frac{d^j}{dt^j}\left(\frac{t}{e^t-1}\right)^a\right\}_{t=0}$$

it follows that $B_k^{(a)}(x)$ is a polynomial in x of degree k. We write $B_k^{(a)}(0) \equiv B_k^{(a)}$. Thus,

$$B_k^{(a)} = \frac{d^k}{dt^k}\left\{\left(\frac{t}{e^t-1}\right)^a\right\}_{t=0}$$

and so $B_k^{(a)}$ is a polynomial in a of degree k.

If $a = 1$, we have the Bernoulli polynomials $B_k^{(1)}(x) \equiv B_k(x)$, and if, further, $x = 0$, we have the Bernoulli numbers $B_k(0) \equiv B_k$. Clearly

$$B_k^{(0)}(x) = x^k \qquad \text{and} \qquad B_0^{(a)}(x) = 1. \tag{2}$$

The first six generalized Bernoulli polynomials are as follows:

$$\begin{aligned}
&B_0^{(a)}(x) = 1, \qquad B_1^{(a)}(x) = x - \frac{a}{2}, \qquad B_2^{(a)}(x) = x^2 - ax + \frac{a(3a-1)}{12},\\
&B_3^{(a)}(x) = x^3 - \frac{3a}{2}x^2 + \frac{a(3a-1)}{4}x - \frac{a^2(a-1)}{8},\\
&B_4^{(a)}(x) = x^4 - 2ax^3 + \frac{a(3a-1)}{2}x^2 - \frac{a^2(a-1)}{2}x\\
&\qquad + \frac{a}{240}(15a^3 - 30a^2 + 5a + 2),\\
&B_5^{(a)}(x) = x^5 - \frac{5a}{2}x^4 + \frac{5a(3a-1)}{6}x^3 - \frac{5a^2(a-1)}{4}x^2\\
&\qquad + \frac{a(15a^3 - 30a^2 + 5a + 2)}{48}x - \frac{a^2(a-1)(3a^2-7a-2)}{96},\\
&B_6^{(a)}(x) = x^6 - 3ax^5 + \frac{5a(3a-1)}{4}x^4\\
&\qquad - \frac{5a^2(a-1)}{2}x^3 + \frac{a(15a^3 - 30a^2 + 5a + 2)}{16}x^2\\
&\qquad - \frac{a^2(a-1)(3a^2-7a-2)}{16}x\\
&\qquad + \frac{a(63a^5 - 315a^4 + 315a^3 + 91a^2 - 42a - 16)}{4032}.
\end{aligned} \tag{3}$$

For a short table of $B_{2k}^{(2\rho)}(\rho)$, see 2.11(17).

Recall that the Bernoulli polynomials are $B_k^{(1)}(x) \equiv B_k(x)$ Thus,

$$\begin{aligned} &B_0(x) = 1, \quad B_1(x) = x - \tfrac{1}{2}, \quad B_2(x) = x^2 - x + \tfrac{1}{6}, \\ &B_3(x) = x^3 - 3x^2/2 + x/2, \quad B_4(x) = x^4 - 2x^3 + x^2 - 1/30, \\ &B_5(x) = x^5 - 5x^4/2 + 5x^3/3 - x/6, \\ &B_6(x) = x^6 - 3x^5 + 5x^4/2 - x^2/2 + 1/42 \end{aligned} \tag{4}$$

The Bernoulli numbers are $B_k \equiv B_k(0)$ Note that $B_k = 0$ when k is odd, $k > 1$ Some additional values of the Bernoulli numbers are

$$\begin{aligned} B_8 = -1/30, \quad B_{10} = 5/66, \quad B_{12} = -691/2730, \\ B_{14} = 7/6, \quad B_{16} = -3617/510 \end{aligned} \tag{5}$$

For further tables of Bernoulli polynomials and numbers, see Fletcher *et al* (1962, pp 65–117) For a detailed study of Bernoulli polynomials and numbers, see Norlund (1954, 1961)

Numerous properties of the polynomials follow directly from the generating formula Thus in (1), replace x by $(a - x)$ and t by $-t$ Then

$$\sum_{k=0}^{\infty} \frac{(-)^k t^k}{k!} B_k^{(a)}(a - x) = \left(\frac{-t}{e^{-t} - 1}\right)^a e^{-t(a-x)} = \frac{t^a e^{xt}}{(e^t - 1)^a}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k^{(a)}(x),$$

and upon equating like powers of t, we have

$$B_k^{(a)}(x) = (-)^k B_k^{(a)}(a - x) \tag{6}$$

Upon differentiating both sides of (1) with respect to t, we are led to the recursion formula

$$aB_k^{(a+1)}(x) = (a - k)B_k^{(a)}(x) + k(x - a)B_{k-1}^{(a)}(x) \tag{7}$$

The following are easily derived from (7)

$$B_k^{(k+1)}(x) = (x - k)B_{k-1}^{(k)}(x) = (-)^k(1 - x)_k \tag{8}$$

$$(x)_k = (-)^{k-1}xB_{k-1}^{(k)}(-x) \tag{9}$$

$$aB_k^{(a+1)}(a) = (a - k)B_k^{(a)}(a) \tag{10}$$

and with the aid of (6),

$$aB_k^{(a+1)}(1) = (a - k)B_k^{(a)} = a(-)^k B_k^{(a+1)}(a) \tag{11}$$

Multiply (1) by itself with a and x replaced by b and y, respectively, and equate like powers of t. Thus,

$$B_k^{(a+b)}(x+y) = \sum_{r=0}^{k} \binom{k}{r} B_{k-r}^{(a)}(y) B_r^{(b)}(x), \tag{12}$$

$$B_k^{(a)}(x+y) = \sum_{k=0}^{k} \binom{k}{r} B_{k-r}^{(a)}(y) x^r, \tag{13}$$

$$\frac{d^m B_k^{(a)}(x)}{dx^m} = \frac{k!}{(k-m)!} B_{k-m}^{(a)}(x). \tag{14}$$

Let $a = k + 1$ and $y = 0$ in (13), and combine with (8) to get

$$(x-1)(x-2)\cdots(x-k) = \sum_{r=0}^{k} \binom{k}{r} B_{k-r}^{(k+1)} x^r, \tag{15}$$

$$(x)_k x^{-k} = \sum_{r=0}^{k-1} (-)^r \binom{k-1}{r} B_r^{(k)} x^{-r}. \tag{16}$$

If we subtract (1) from itself with x replaced by $x + 1$, then

$$\Delta B_k^{(a)}(x) = k B_{k-1}^{(a-1)}(x), \qquad \Delta^m B_k^{(a)}(x) = \frac{k!}{(k-m)!} B_{k-m}^{(a-m)}(x), \tag{17}$$

where Δ is the forward difference operator with respect to x. Hence,

$$B_k^{(a)}(x+y) = \sum_{r=0}^{k} \binom{k}{r} B_{k-r}^{(a-r)}(y) x(x-1)\cdots(x-r+1), \tag{18}$$

$$x^k = \sum_{r=0}^{k} \binom{k}{r} B_{k-r}^{(-r)} x(x-1)\cdots(x-r+1). \tag{19}$$

To obtain a representation for $B_k^{(a)}(x)$ when a is a negative integer, replace k by $k + m$ in (17) and set $a = 0$. Thus,

$$B_k^{(-m)}(x) = \frac{k!}{(k+m)!} \Delta^m x^{k+m} = \frac{k!}{(k+m)!} \sum_{s=0}^{m} (-)^r \binom{m}{s} (x+m-s)^{k+m}. \tag{20}$$

From (14) with $m = 1$ and k replaced by $(k + 1)$, we have

$$B_{k+1}^{(a)}(x) = (k+1) \int_0^x B_k^{(a)}(t)\, dt + B_{k+1}^{(a)}. \tag{21}$$

This is very convenient to generate the Bernoulli polynomials when a is independent of x, provided that the numbers $B_{k+1}^{(a)}$ are readily available We now establish the recurrence formula

$$B_{k+1}^{(a)}(x) = B_1^{(a)}(x)B_k^{(a)}(x) - a \sum_{r=0}^{[(k-1)/2]} \frac{\binom{k}{2r+1} B_{2r+2}}{(2r+2)} B_{k-2r-1}^{(a)}(x) \tag{22}$$

Let

$$g(t) = \frac{t^a}{(e^t - 1)^a} e^{xt}$$

Then upon taking the logarithm of both sides and differentiating, we get

$$g'(t) = g(t)h(t),$$

$$h(t) = x + \frac{a}{t}\left(1 - \frac{t}{1 - e^{-t}}\right) = B_1^{(a)}(x) - a \sum_{r=0}^{\infty} \frac{t^{2r+1}B_{2r+2}}{(2r+2)!},$$

since $B_{2r+1} = 0$ if $r > 1$ Now

$$h(0) = B_1^{(a)}(x), \qquad h^{(2s)}(0) = 0 \qquad \text{for} \quad s > 0,$$

$$h^{(2s+1)}(0) = -(aB_{2s+2})\,(2s+2) \qquad g^{(s)}(0) = B_s^{(a)}(x),$$

and upon substituting these quantities into

$$g^{(k+1)}(0) = \sum_{r=0}^{k} \binom{k}{r} h^{(r)}(0)g^{(k-r)}(0),$$

which is Leibnitz's rule, we get (22) As a corollary, we see that when $a = 2x$, $B_1^{(2x)}(x) = 0$ and so by induction $B_{2k+1}^{(2x)}(x) = 0$ for $k = 0, 1, 2, \ldots$, a result which also follows from (6)

From (21) and (17),

$$\int_x^y B_k^{(a)}(t)\,dt = [B_{k+1}^{(a)}(y) - B_{k+1}^{(a)}(x)]\,(k+1), \tag{23}$$

$$\int_x^{x+1} B_k^{(a)}(t)\,dt = B_k^{(a-1)}(x) \tag{24}$$

In particular, if $a = 1$,

$$B_k(x+1) - B_k(x) = kx^{k-1}, \tag{25}$$

$$\int_x^{x+1} B_k(t)\,dt = x^k \tag{26}$$

Thus from (25) and (6),

$$B_k(1) = B_k = (-)^k B_k, \qquad k \geqslant 2, \tag{27}$$

which shows that except for B_1, the Bernoulli numbers of odd index are zero. It follows from (26) and (23) that

$$\sum_{r=0}^{m-1} r^k = \int_0^m B_k(t)\,dt = [B_{k+1}(m) - B_{k+1}]/(k+1). \tag{28}$$

The Bernoulli polynomials may be expressed as a Fourier series. Consider

$$\int_C f(z)\,dz, \qquad f(z) = z^{-k}e^{xz}(e^z - 1)^{-1}, \qquad k > 0,$$

where C is a large circle with center at the origin and radius $(2N+1)\pi$, N an integer. The points $z_r = 2\pi ir$, $r = 0, \pm 1, \pm 2, \ldots$ are poles of $f(z)$, and the residue of $f(z)$ at z_r, $r \neq 0$, is $(2\pi ir)^{-k}e^{2\pi irx}$. From (1) with $a = 1$, the residue at $z = 0$ is $B_k(x)/k!$. So long as $0 \leqslant x \leqslant 1$, the integral around C tends to zero as $N \to \infty$. It follows that

$$B_{2k}(x) = \frac{2(-)^{k+1}(2k)!}{(2\pi)^{2k}} \sum_{r=1}^{\infty} r^{-2k} \cos 2\pi rx, \qquad k > 0,\ 0 \leqslant x \leqslant 1, \tag{29}$$

$$B_{2k+1}(x) = \frac{2(-)^{k+1}(2k+1)!}{(2\pi)^{2k+1}} \sum_{r=1}^{\infty} r^{-2k-1} \sin 2\pi rx, \qquad k > 0,\ 0 \leqslant x \leqslant 1;$$

$$k = 0,\ 0 < x < 1. \tag{30}$$

Thus,

$$B_{2k+1} = 0, \qquad B_{2k} = \frac{2(-)^{k+1}(2k)!}{(2\pi)^{2k}} \sum_{r=1}^{\infty} r^{-2k}, \qquad k > 0. \tag{31}$$

From (1), we readily deduce that

$$z \cot z = \sum_{n=0}^{\infty} \frac{(-)^n 2^{2n} B_{2n}}{(2n)!} z^{2n}, \qquad |z| < \pi, \tag{32}$$

$$\frac{\tan z}{z} = \sum_{n=0}^{\infty} \frac{(-)^n 2^{2n+2}(2^{2n+2} - 1)B_{2n+2}}{(2n+2)!} z^{2n}, \qquad |z| < \pi/2, \tag{33}$$

$$z \csc z = 2 \sum_{n=0}^{\infty} \frac{(-)^n (1 - 2^{2n-1})B_{2n}}{(2n)!} z^{2n}, \qquad |z| < \pi, \tag{34}$$

$$\ln \cos z = \sum_{n=1}^{\infty} \frac{(-)^n 2^{2n-1}(2^{2n} - 1)B_{2n}}{(2n)!\,n} z^{2n}, \qquad |z| < \pi/2. \tag{35}$$

2.9. The D and δ Operators

These convenient symbols are defined by

$$\delta = zD, \qquad D = d/dz \tag{1}$$

The following elementary properties are easily proved

$$\delta_x = \nu^{-1}\delta_z \qquad \delta_z = z\,d/dz, \quad z = ax^\nu \tag{2}$$

$$(\delta + a)z^\nu = z^\nu(\nu + a), \qquad \prod_{i=1}^{p}(\delta + a_i)z^\nu = z^\nu \prod_{i=1}^{p}(\nu + a_i), \tag{3}$$

$$\prod_{h=1}^{n}(\delta + \nu + h - 1)f(z) = z^{1-\nu}D^n\{z^{\nu+n-1}f(z)\} \tag{4}$$

Thus from (3)

$$\delta(\delta - 1) \quad (\delta - n + 1)z^\nu = z^\nu\nu(\nu - 1) \quad (\nu - n + 1) = z^n D^n z^\nu$$

which implies that

$$z^n D^n = \delta(\delta - 1) \quad (\delta - n + 1) \tag{5}$$

The right-hand side of (5) may be interpreted as an algebraic quantity, and with the aid of 2.8(15), we have

$$z^n D^n = \sum_{r=0}^{n}\binom{n-1}{r-1}B_{n-r}^{(n)}\delta^r = \sum_{r=0}^{n}\binom{n-1}{r}B_r^{(n)}\delta^{n-r} \tag{6}$$

Using 2.8(19) and (5), we get

$$\delta^n = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{(-k)}z^k D^k = \sum_{k=0}^{n}\binom{n}{k}B_k^{(k-n)}z^{n-k}D^{n-k} \tag{7}$$

Thus,

$$\begin{aligned}
&\delta = zD, \qquad \delta^2 = z^2D^2 + zD, \qquad \delta^3 = z^3D^3 + 3z^2D^2 + zD,\\
&\delta^4 = z^4D^4 + 6z^3D^3 + 7z^2D^2 + zD,\\
&\delta^5 = z^5D^5 + 10z^4D^4 + 25z^3D^3 + 15z^2D^2 + zD,\\
&\delta^6 = z^6D^6 + 15z^5D^5 + 65z^4D^4 + 90z^3D^3 + 31z^2D^2 + zD
\end{aligned} \tag{8}$$

To compute higher powers of δ, write

$$\delta^n = \sum_{k=0}^{n-1} b_k^{(n)} z^{n-k}D^{n-k}, \qquad b_k^{(n)} = \binom{n}{k}B_k^{(k-n)} \tag{9}$$

Then operating on both sides of this equation by δ, we find the recursion formula

$$b_k^{(n+1)} = b_k^{(n)} + (n+1-k)b_{k-1}^{(n)}, \qquad b_0^{(n)} = b_{n-1}^{(n)} = 1, \quad b_k^{(n)} = 0 \quad \text{for} \quad k > n. \tag{10}$$

It can be readily shown by induction that

$$\begin{aligned}
\prod_{i=1}^{p} (\delta + a_i) &= \sum_{m=0}^{p} c_{p,m} z^m D^m, \qquad c_{p,m} = \sum_{t=0}^{p-m} \binom{p-t}{p-t-m} B_{p-t-m}^{(-m)} S_t(a_p), \\
c_{p,p} &= 1, \qquad c_{p,p-1} = \frac{p(p-1)}{2} + S_1(a_p), \\
c_{p,p-2} &= \frac{p(p-1)(p-2)(3p-5)}{24} + \frac{(p-1)(p-2)}{2} S_1(a_p) + S_2(a_p), \\
c_{p,p-3} &= \frac{p(p-1)(p-2)^2(p-3)^2}{48} \\
&\quad + \frac{(p-1)(p-2)(p-3)(3p-8)}{24} S_1(a_p) \\
&\quad + \frac{(p-2)(p-3)}{2} S_2(a_p) + S_3(a_p), \\
&\ \ \vdots \\
c_{p,0} &= S_p(a_p),
\end{aligned} \tag{11}$$

where $S_0(a_p) = 1$, and for $m > 0$, $S_m(a_p)$ are the symmetric polynomials

$$\begin{aligned}
S_m(a_p) &= \sum a_{t_m} a_{t_{m-1}} \cdots a_{t_1}, \qquad t_m > t_{m-1} > \cdots > t_1, \\
&\qquad t_j \in \{1, 2, \ldots, p\}, \qquad j = 1, 2, \ldots, m.
\end{aligned} \tag{12}$$

These polynomials may also be implicitly defined by

$$\prod_{i=1}^{p} (x + a_i) = \sum_{m=0}^{p} S_m(a_p)\, x^{p-m}. \tag{13}$$

We can give an alternative representation for the coefficients $c_{p,m}$ in (11). Let $\prod_{i=1}^{p} (\delta + a_i)$ operate on z^x, x independent of z. Then

$$\prod_{i=1}^{p} (x + a_i) = \sum_{m=0}^{p} c_{p,m} x(x-1) \cdots (x-m+1),$$

$$c_{p,m} = (1/m!)\, \Delta^m \left\{ \prod_{i=1}^{p} (x + a_i) \right\}_{x=0}.$$

Since

$$(1/m!)\,\Delta^m\left\{\prod_{i=1}^{p}(x+a_i)\right\}_{x=0} = (1/m!)\,\Delta^m\left\{\sum_{j=0}^{p} S_{p-j}(a_p)x^j\right\}_{x=0}$$

$$= 1/m!\sum_{j=0}^{p} S_{p-j}(a_p)\left\{(-)^m\sum_{r=0}^{m}[(-m)_r/r!]r^j\right\}$$

in view of 2.8(20), we have

$$c_{p,m} = \frac{(-)^m}{m!}\sum_{r=0}^{m}\frac{(-m)_r}{r!}\sum_{j=0}^{p} a_{p-j}r^j = \frac{(-)^m}{m!}\sum_{r=0}^{m}\frac{(-m)_r}{r!}\prod_{i=1}^{p}(r+a_i),$$

$$c_{p,m} = \frac{(-)^m}{m!}\left(\prod_{i=1}^{p} a_i\right){}_{p+1}F_p\left(\begin{matrix}-m, 1+a_p\\ a_p\end{matrix}\middle|\,1\right), \tag{14}$$

in the notation of 3.2(1). Since $\Delta^m x^p = 0$ for $m > p$,

$${}_{p+1}F_p\left(\begin{matrix}-m, 1+a_p\\ a_p\end{matrix}\middle|\,1\right) = 0, \qquad m > p \tag{15}$$

2.10 Power Series and Other Expansions

2.10.1 Power Series Expansions

From the second form of 2.4(3), we have

$$\psi(z+1) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{z+k}\right) \tag{1}$$

and so by differentiation

$$\psi^{(n)}(z+1) = d^n\psi(z+1)/dz^n = (-)^{n+1}n!\sum_{k=1}^{\infty}(z+k)^{-n-1}, \qquad n > 0 \tag{2}$$

Thus

$$\ln\Gamma(z+1) = \sum_{k=1}^{\infty}[(-)^k S_k z^k/k], \qquad |z| < 1,$$

$$S_1 = -\psi(1) = \gamma, \qquad S_k = \sum_{r=1}^{\infty} r^{-k}, \qquad k = 2, 3, \ldots, \tag{3}$$

$$\psi(z+1) = \sum_{k=1}^{\infty}(-)^k S_k z^{k-1}, \qquad |z| < 1 \tag{4}$$

For references to tables of S_k, see Fletcher *et al.* (1962, Vol. 1, p. 84). We note from 2.8(31) that

$$S_{2k} = \frac{(-)^{k+1}(2\pi)^{2k}B_{2k}(0)}{2(2k)!}, \qquad k > 0. \tag{5}$$

Thus

$$\psi'(1) = S_2 = \pi^2/6, \qquad S_4 = \pi^4/90, \qquad S_6 = \pi^6/945, \quad \text{etc.} \tag{6}$$

The function S_k is a special case of the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \qquad R(z) > 1. \tag{7}$$

Let

$$1/\Gamma(z+1) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_0 = 1, \quad |z| < \infty. \tag{8}$$

Take the logarithmic derivative of (8) and compare with (3). Then

$$\sum_{k=0}^{\infty} (k+1)a_{k+1}z^k = \left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{k=0}^{\infty} (-)^k S_{k+1} z^k\right),$$

and upon equating like powers of z, we find

$$ra_r = \sum_{k=1}^{r} (-)^{k+1} S_k a_{r-k}. \tag{9}$$

In a similar fashion, if

$$\Gamma(z+1) = \sum_{k=0}^{\infty} b_k z^k, \qquad b_0 = 1, \quad |z| < 1, \tag{10}$$

then

$$rb_r = -\sum_{k=1}^{r} (-)^{k+1} S_k b_{r-k}. \tag{11}$$

Clearly

$$\sum_{k=0}^{r} a_k b_{r-k} = 0, \qquad r \geqslant 1. \tag{12}$$

From (10),

$$\Gamma(z+1) - \sum_{r=1}^{m} 1/(z+r) = \sum_{k=0}^{\infty}\left[b_k - (-)^k \sum_{r=1}^{m} r^{-k-1}\right] z^k, \quad |z| < m \tag{13}$$

The coefficients a_k and the Taylor series coefficients for $\Gamma(z+3)$, which are easily derived from the a_k's, are given to $20d$ in Chapter XVII, Table 6 These have been rounded from $25d$ tables which were developed using the $32d$ values of S_k given by Peters (1957) The latter $25d$ tables were then employed with the aid of 8 4 2(15) to get coefficients for the expansion of $\Gamma(x+3)$ and the reciprocal of $\Gamma(x+1)$ in series of the shifted Chebyshev polynomials of the first kind, valid for $0 \leqslant x \leqslant 1$ These coefficients rounded to $20d$ are presented in Chapter XVII, Tables 8 and 7, respectively

2 10 2 Chebyshev Expansions for $\ln \Gamma(z)$ and Its Derivatives

Here we apply the developments of 8 4 1(19, 20) [see also Wimp (1961)] to derive expansions for $\ln \Gamma(z)$ and its derivatives in series of shifted Chebyshev polynomials of the first kind From 2 5(1),

$$\psi^{(m)}(z) = (-)^{m+1} \int_0^{\infty} e^{-zt} t^m (1 - e^{-t})^{-1}\, dt \qquad R(z) > 0, \quad m > 0, \tag{1}$$

so that if

$$\psi^{(m)}(x+a) = \sum_{n=0}^{\infty} C_n^{(m)} T_n^*(x), \qquad 0 \leqslant x \leqslant 1, \tag{2}$$

$$C_n^{(m)} = \epsilon_n \exp[i\pi(n - \tfrac{1}{2})/2](-2)^{m+1} \mathscr{H}\{t^{m-1} \exp[-(2a+1)t](1 - e^{-2t})^{-1}, e^{i\pi/2}, n\} \tag{3}$$

Since

$$(1 - e^{-2t})^{-1} = \sum_{k=0}^{N-1} e^{-2kt} + (1 - e^{-2t})^{-1} e^{-2Nt}, \tag{4}$$

we find that $C_n^{(m)}$ can be represented by the infinite series

$$C_n^{(m)} = -2(-)^n \epsilon_n \sum_{k=0}^{\infty} f_n^{(m)}(a), \qquad n + m > 0, \tag{5}$$

where

$$\begin{aligned} f_n^{(m)}(a) &= \frac{d^m}{da^m}\left\{\frac{[p + (p^2-1)]^{-n}}{(p^2-1)^{1/2}}\right\}, \qquad p = 2k + 2a + 1, \\ &= \frac{(-)^m (m+n)!\, 2^m e^{-n\theta}}{n!\, \sinh^{m+1}\theta}\, {}_2F_1\left(\begin{matrix} -m & m+1 \\ n+1 & \end{matrix}\middle|\, \frac{1 - \coth\theta}{2}\right), \\ &\qquad p = \cosh\theta \end{aligned} \tag{6}$$

The coefficients $C_n^{(m)}$ are tabulated in Chapter XVII, Table 8, for $a = 3$ and $m = 0(1)6$. We also give coefficients for $\ln \Gamma(x + 3)$. It is of interest to briefly describe how these coefficients were obtained. It is readily seen that the series in (5) converges very slowly for n and m small. Even for moderate n and m the convergence is not too rapid. However, the situation can be remedied in part by use of the Euler–Maclaurin summation formula [see Steffensen (1950)]. In this manner we computed $C_n^{(m)}$ for $m = 3$ and $n = 6(1)26$ to an accuracy of about $28d$. To evaluate $C_n^{(m)}$ for $m = 3$ and $n = 0(1)5$, we made use of known values of $\psi^{(3)}(z)$ and its integrals to write six equations involving the above six coefficients. The coefficients $C_n^{(m)}$ for $m = 0, 1, 2$ and the coefficients for $\ln \Gamma(x + a)$ were evaluated by successive integration of (2) with the aid of 8.6.1(7). The coefficients $C_n^{(m)}$ for $m = 4, 5$, and 6 were determined by differentiation of (2) [see 8.6.1(6)]. In all the computations $28d$ were carried. The integration process produced essentially no loss in the accuracy. However, each step of the differentiation process produced a loss of about three decimals. The coefficients were rounded to $20d$ for presentation in Chapter XVII, Table 8.

2.10.3. An Expansion for $\Gamma(z + 1)$

The following development is due to Lanczos (1964). We start with 2.1(1), replace z by $z + \frac{1}{2}$ and t by $(z + \sigma + \frac{3}{2})(1 - \ln v)$ where σ is arbitrary except as noted below and v is the new variable of integration. Thus,

$$\begin{aligned} \Gamma(z + \tfrac{1}{2}) &= (z + \sigma + \tfrac{1}{2})^{z+\frac{1}{2}} \exp[-(z + \sigma + \tfrac{1}{2})]\, F(z), \\ F(z) &= \int_0^e [v(1 - \ln v)]^{z-\frac{1}{2}} v^\sigma \, dv, \quad R(z + \sigma + \tfrac{1}{2}) > 0. \end{aligned} \tag{1}$$

Next, introduce the transformation

$$v(1 - \ln v) = \cos^2 \theta, \qquad dv/d\theta = \sin 2\theta / \ln v \tag{2}$$

where $v = 0$, 1, and e correspond to $\theta = -\pi/2$, 0, and $\pi/2$, respectively. Thus,

$$F(z) = \int_{-\pi/2}^{\pi/2} \cos^{2z} \theta f(\theta)\, d\theta, \qquad f(\theta) = (2v^\sigma \sin \theta)/\ln v, \quad R(z + \sigma + \tfrac{1}{2}) > 0. \tag{3}$$

To compute $F(z)$ in an efficient manner, expand $f(\theta)$ in a Fourier cosine series. This series is simplified since the transcendental equation in (2) implies that v can be written as a convergent power series in $\cos^2 \theta = (1 + \cos 2\theta)/2$, so that only cosine terms of even index appear.

In this Fourier series, express $\cos 2k\theta$ in powers of $\cos\theta$ [see 8 5 1(7)] and then use 2 6(5) to get

$$\Gamma(z+1) = (2\pi)^{1/2}(z+\sigma+\tfrac{1}{2})^{z+\frac{1}{2}}\exp[-(z+\sigma+\tfrac{1}{2})]\sum_{k=0}^{\infty} g_k H_k(z),$$

$$R(z+\sigma+\tfrac{1}{2}) > 0, \tag{4}$$

$$g_k = (2\pi)^{-1/2}e^{\sigma}\epsilon_k(-)^k\sum_{r=0}^{k}(-)^r\binom{k}{r}(k)_r\left(\frac{e}{r+\sigma+\frac{1}{2}}\right)^{r+\frac{1}{2}},$$

$$\epsilon_0 = 1, \qquad \epsilon_k = 2 \qquad \text{for} \quad k = 1, 2, \ldots, \tag{5}$$

$$H_0(z) = 1, \qquad H_k(z) = [(z+1)_k(z+1)_{-k}]^{-1}$$

$$= \frac{(-)^k(-z)_k}{(z+1)_k} = \frac{z(z-1)\cdots(z-k+1)}{(z+1)(z+2)\cdots(z+k)} \tag{6}$$

We also have

$$\frac{H_k(z+1)}{H_k(z)} = \frac{(z+1)^2}{(z+1)^2-k^2}, \qquad \frac{H_{k+1}(z)}{H_k(z)} = \frac{z-k}{z+k+1} \tag{7}$$

Note that if z is a positive integer, (4) terminates in view of (6) and we find

$$\sum_{k=0}^{n}\frac{(-)^k(-n)_k}{(n+1)_k}g_k = \frac{(2\pi)^{-1/2}\exp(n+\sigma+\frac{1}{2})n!}{(n+\sigma+\frac{1}{2})^{n+\frac{1}{2}}} \tag{8}$$

which affords an alternative method for the evaluation of the g_k's Indeed, in practice we found (8) more efficient than (5) From the asymptotic nature of $\Gamma(z)$ for z large, we deduce that

$$\sum_{k=0}^{\infty} g_k = 1 \tag{9}$$

If in (4), we put $z = n - \frac{1}{2}$, then

$$\sum_{k=0}^{\infty}[(n+\tfrac{1}{2})_k(n+\tfrac{1}{2})_{-k}]^{-1}g_k = [2^{-1/2}(\tfrac{1}{2})_n e^{n+\sigma}](n+\sigma)^n, \tag{10}$$

and for $n = 0$ and $n = 1$, we have the respective equations

$$\sum_{k=0}^{\infty}(-)^k g_k = 2^{-1/2}e^{\sigma} \tag{11}$$

$$\sum_{k=0}^{\infty}\frac{(-)^{k+1}g_k}{4k^2-1} = \frac{2^{1/2}e^{\sigma+1}}{4(\sigma+1)}, \tag{12}$$

which are useful for checking numerical values of the g_k's

From the theory of Fourier series [see, for example, Zygmund (1959)], the series for $F(z)$ and hence also for $\Gamma(z+1)$, converges for all z, $R(z+\sigma+\frac{1}{2})>0$. Notice that the infinite series portion of (4) mimics the poles of $\Gamma(z+1)$. That the infinite series is slowly convergent is to be expected for though $\sin\theta/\ln v$ [see (3)] has a limit when $\theta\to-\pi/2$, its derivative there is infinite. The role of σ in the expression for $f(\theta)$ is to smooth out this irregularity. Thus the larger we take $\sigma>0$, the smoother $f(\theta)$ and some of its higher derivatives. The magnitude of the coefficients g_k increase with σ and more terms in the series are required before reaching the stage where the g_k's fail to diminish rapidly. If we truncate the series nearly where the g_k's begin to level off, we can expect to achieve good accuracy in the right half-plane. Now the maximum error incurred when $f(\theta)$ is approximated by n terms of its Fourier series, call it $f_n(\theta)$, occurs near $\theta=-\pi/2$, since for this value of $\theta, f(\theta)$ fails to be analytic. Thus,

$$r_n=\max|f(\theta)-f_n(\theta)|\sim 2^{1/2}\rho_n\,,\qquad \rho_n=2^{-1/2}e^{\sigma}-\sum_{k=0}^{n-1}(-)^k g_k\,. \tag{13}$$

If $S_n(z)$ is the error in the infinite series portion of (4) truncated after n terms, we find that

$$\begin{aligned}\Gamma(z+1)&=(2\pi)^{1/2}(z+\sigma+\tfrac{1}{2})^{z+\frac{1}{2}}\exp[-(z+\sigma+\tfrac{1}{2})]\left[\sum_{k=0}^{n-1}g_kH_k(z)+S_n(z)\right],\\ &\qquad\qquad R(z+\sigma+\tfrac{1}{2})>0,\\ |S_n(z)|\leqslant U_n(z),\qquad U_n(z)&\sim V_n(z)=\pi\left|\frac{\rho_n\Gamma(z+1)}{\Gamma(z+\frac{1}{2})}\right|\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)},\\ &\qquad\qquad x=R(z)>0.\end{aligned} \tag{14}$$

This estimate is found to be quite conservative in practice. For $\sigma=5$, the coefficients g_k are recorded to $20d$ in Chapter XVII, Table 9. There we also tabulate ρ_n and offer other comments on the accuracy of the approximation to $\Gamma(z+1)$.

2.11. Asymptotic Expansions

If we combine 2.5(5) and 2.8(1), and use Watson's lemma (see 1.4), then with the aid of 2.1(3), we get

$$\begin{aligned}\ln\Gamma(z)&=(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln(2\pi)\\ &\quad+\sum_{k=1}^{n}B_{2k}[(2k-1)(2k)z^{2k-1}]^{-1}+O(z^{-2n-1}),\\ &\qquad\qquad |\arg z|\leqslant\pi-\epsilon,\qquad \epsilon>0.\end{aligned} \tag{1}$$

This is equivalent to

$$\Gamma(z) = e^{-z}z^{z-\frac{1}{2}}(2\pi)^{1/2}\left[1 + \frac{z^{-1}}{12} + \frac{z^{-2}}{288} - \frac{139z^{-3}}{51840} + O(z^{-4})\right],$$
$$|\arg z| \leqslant \pi - \epsilon, \qquad \epsilon > 0, \tag{2}$$

or

$$\Gamma(z) = (2\pi)^{1/2}\left[\exp\left\{-z + (z - \tfrac{1}{2})\ln z + \frac{z^{-1}}{12}\right\}\right]\left[1 - \frac{z^{-2}}{360} + O(z^{-4})\right],$$
$$|\arg z| \leqslant \pi - \epsilon, \qquad \epsilon > 0 \tag{3}$$

Barnes (1899) and Rowe (1931) have shown that

$$\ln \Gamma(z + a) = (z + a - \tfrac{1}{2})\ln z - z + \tfrac{1}{2}\ln 2\pi$$
$$+ \sum_{k=1}^{n} \frac{(-)^{k+1}B_{k+1}(a)}{k(k+1)} z^{-k} + O(z^{-n-1}), \tag{4}$$
$$|\arg z| \leqslant \pi - \epsilon, \qquad \epsilon > 0.$$

If $a = \frac{1}{2}$, $B_k(a)$ vanishes if k is odd, see 2 8(6) and

$$\ln \Gamma(z + \tfrac{1}{2}) = z(\ln z - 1) + \tfrac{1}{2}\ln 2\pi + \sum_{k=1}^{n} \frac{B_{2k}(\frac{1}{2})}{2k(2k-1)} z^{1-2k} + O(z^{-2n-1}),$$
$$|\arg z| \leqslant \pi - \epsilon, \qquad \epsilon > 0 \tag{5}$$

An elementary proof of (4) communicated to me by J L Fields follows From (1), if a is bounded, we can write

$$\ln \Gamma(z + a) \sim (z + a - \tfrac{1}{2})\ln z - z + \tfrac{1}{2}\ln 2\pi + (z + a - \tfrac{1}{2})\ln\left(1 + \frac{a}{z}\right) - a$$
$$+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k-1)(2k)(z+a)^{2k-1}},$$

$$\ln \Gamma(z + a) \sim (z + a - \tfrac{1}{2})\ln z - z + \tfrac{1}{2}\ln 2\pi + \sum_{k=1}^{\infty} \frac{(-)^{k+1}P_{k+1}(a)}{k(k+1)} z^{-k}$$

uniformly in a, a near zero, for $|\arg(z + a)| \leqslant \pi - \epsilon$, $\epsilon > 0$, where to complete the proof, $P_k(a)$ must be identified as $B_k(a)$ Now

$$\psi(z + a) = (d/dz)\ln \Gamma(z + a) = (d/da)\ln \Gamma(z + a)$$

Performing the indicated operations and equation like powers of z, we have $P_0(a) = 1$, $P_1(a) = a - \frac{1}{2} = B_1(a)$ and

$$(d/da)P_k(a) = kP_{k-1}(a)$$

Since $P_k(0) = B_k$, it follows by induction from 2.8(14) that $P_k(a) = B_k(a)$. We have the limiting forms,

$$\lim_{z\to\infty} e^z z^{\frac{1}{2}-z}\Gamma(z) = (2\pi)^{1/2}, \qquad |\arg z| \leqslant \pi - \epsilon, \quad \epsilon > 0, \tag{6}$$

$$\lim_{|y|\to\infty} e^{\frac{1}{2}\pi|y|}|y|^{\frac{1}{2}-x}|\Gamma(x+iy)| = (2\pi)^{1/2}, \qquad x, y \text{ real}. \tag{7}$$

Differentiating (1), (4), and (5), respectively, gives

$$\psi(z) = \ln z - (2z)^{-1} - \sum_{k=1}^{n-1} B_{2k}z^{-2k}/2k + O(z^{-2n}), \qquad |\arg z| \leqslant \pi - \epsilon, \quad \epsilon > 0, \tag{8}$$

$$\psi(z+a) = \ln z - \sum_{k=0}^{n-1} \frac{(-)^{k+1}B_{k+1}(a)}{k+1} z^{-k-1} + O(z^{-n-1}), \qquad |\arg z| \leqslant \pi - \epsilon, \quad \epsilon > 0, \tag{9}$$

$$\psi(z+\tfrac{1}{2}) = \ln z - \sum_{k=0}^{n-1} \frac{B_{2k+2}(1/2)}{2k+2} z^{-2k-2} + O(z^{-2n-2}), \qquad |\arg z| \leqslant \pi - \epsilon, \quad \epsilon > 0. \tag{10}$$

From (2),

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + \frac{(a-b)(a+b-1)}{2z} + \frac{(a-b)(a-b-1)}{24z^2} \times \{3(a+b-1)^2 - a + b - 1\}\right][1 + O(z^{-3})], \qquad |\arg z| \leqslant \pi - \epsilon, \quad \epsilon > 0. \tag{11}$$

We now present a generalization of (11) due to Tricomi and Erdélyi (1951) [see also Nörlund (1961)]. Thus,

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\sum_{k=0}^{N-1} \frac{(-)^k B_k^{(a-b+1)}(a)(b-a)_k}{k!} z^{-k} + z^{a-b}O(z^{-N}), \qquad |\arg(z+a)| \leqslant \pi - \epsilon, \quad \epsilon > 0, \tag{12}$$

where a and b are bounded complex numbers and $B_k^{(a-b+1)}(a)$ is the generalized Bernoulli polynomial (see 2.8). If $b - a = 1 - N$, then (12) is exact. That is, we can ignore the $O(z^{-N})$ term and the restriction on $\arg z$. Further, if $b - a$ is a positive integer m, and $z > \max\{|a|, |b-1|\}$, then with $N \to \infty$ the power series is convergent and sums to $[(z+a)(z+a+1)\cdots(z+a+m-1)]^{-1}$. Thus in this case also, the $O(z^{-N})$ term and the restriction on $\arg z$ can be omitted. A convergent factorial expansion for $\Gamma(z+a)/\Gamma(z+b)$ of the same general character as (12) has been given by Nörlund [1961, Eq. (43)].

For the proof of (12) we use 2.7(11) with $x = z + a$ and $y = b - a$. Then with the aid of 2.6(3) and 2.2(2),

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{\Gamma(a+1-b)}{2\pi i}\int_{-\infty e^{i\delta}}^{(0+)} e^{vz} f(v)\,dv, \qquad f(v) = e^{av}(e^v - 1)^{b-a-1},$$
$$-\pi/2 < \delta < \pi/2 \qquad \delta - \pi \leqslant \arg(e^v - 1) \leqslant \delta + \pi, \tag{13}$$
$$\arg(z+a) \leqslant \pi - \epsilon, \qquad \epsilon > 0, \qquad b - a \neq 1, 2, 3,$$

Now use 2.8(1) to represent $f(v)$. From the discussion in 1.4, Watson's lemma is applicable to the loop integral in (13), and termwise integration with the aid of 2.7(2) leads to (12).

Recently, Fields (1966) has shown that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = (z+a-\rho)^{a-b}\sum_{k=0}^{N-1}\frac{B_{2k}^{(2\rho)}(\rho)(b-a)_{2k}(z+a-\rho)^{-2k}}{(2k)!}$$
$$+ (z+a-\rho)^{a-b}O((z+a-\rho)^{-2N}), \tag{14}$$
$$2\rho = 1 + a - b \qquad |\arg(z+a)| \leqslant \pi - \epsilon, \qquad \epsilon > 0,$$

where $B_{2k}^{(2\rho)}(\rho)$ is the generalized Bernoulli polynomial (see 2.8). Note that this series is essentially an even one. The proof readily follows from (13), which we rewrite in the form

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{\Gamma(2\rho)}{2\pi i}\int_{\infty e^{i\delta}}^{(0+)} e^{(z+a-\rho)v} v^{-2\rho} h(v)\,dv,$$
$$h(v) = e^{v\rho} v^{2\rho}(e^v - 1)^{-2\rho} = \left[\frac{2\sinh v/2}{v}\right]^{-2\rho}, \tag{15}$$

with the same conditions as in (13). Next employ 2.8(1) to represent $h(v)$ and (14) follows upon application of Watson's lemma and termwise integration.

A short enumeration of $B_{2k}^{(2\rho)}(\rho)$, which are polynomials in ρ of degree k, follows

$$B_0^{(2\rho)}(\rho) = 1, \qquad B_2^{(2\rho)}(\rho) = -\frac{\rho}{6} \qquad B_4^{(2\rho)}(\rho) = \frac{\rho(5\rho+1)}{60},$$
$$B_6^{(2\rho)}(\rho) = -\frac{\rho}{504}(35\rho^2 + 21\rho + 4),$$
$$B_8^{(2\rho)}(\rho) = \frac{\rho}{2160}(175\rho^3 + 210\rho^2 + 101\rho + 18) \tag{16}$$
$$B_{10}^{(2\rho)}(\rho) = -\frac{\rho}{3168}[385\rho^4 + 770\rho^3 + 671\rho^2 + 286\rho + 48],$$
$$B_{12}^{(2\rho)}(\rho) = \frac{\rho}{7\,86240}[1\,75175\rho^5 + 5\,25525\rho^4 + 7\,15715\rho^3 + 5\,31531\rho^2$$
$$+ 2\,07974\rho + 33168]$$

To generate further polynomials, use

$$B_{2k}^{(2\rho)}(\rho) = -2\rho \sum_{r=0}^{k-1} \frac{\binom{2k-1}{2r+1} B_{2r+2}}{2r+2} B_{2k-2r-2}^{(2\rho)}(\rho), \tag{17}$$

which is readily derived from 2.8(22).

The formulas (12) and (14) are valid even though $\Gamma(z+a)/\Gamma(z+b)$ has poles at $z = -a - n$, n a positive integer or zero, if $|z|$ is sufficiently large, $|\arg(z+a)| < \pi$. But the approximation will break down when used for moderate values of z if z is near one of these poles. To obviate this difficulty, use the reflection formula for gamma functions to write

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{\sin \pi(z+b)}{\sin \pi(z+a)} \frac{\Gamma(1-z-b)}{\Gamma(1-z-a)},$$

and then use the asymptotic expansion for the ratio of the gamma functions on the right of the latter equation for $|\arg(-z)| \leqslant \pi - \epsilon$, $\epsilon > 0$.

With $z = n$, $a = -x$, and $b = 1$, we get from (12) and (14) useful expressions for the binomial coefficient. Thus (12) gives

$$\binom{x}{n} \sim \frac{(-)^n n^{-(x+1)}}{\Gamma(-x)} \sum_{k=0}^{\infty} \frac{(x+1)_k B_k^{(-x)}}{k!\, n^k},$$

$$\binom{x}{n} \sim \frac{(-)^n n^{-(x+1)}}{\Gamma(-x)} \Big[1 + \frac{(x)_2}{2n} + \frac{(x)_3(3x+1)}{24n^2} + \frac{(x)_4(x)_2}{48n^3} + \frac{(x)_5}{5760n^4}(15x^3 + 30x^2 + 5x - 2) + \cdots\Big], \tag{18}$$

and from (14),

$$\binom{x}{n} \sim \frac{(-)^n (n - x/2)^{-(x+1)}}{\Gamma(-x)} \sum_{k=0}^{\infty} \frac{B_{2k}^{(2\rho)}(\rho)(x+1)_{2k}}{(2k)!\,(n - x/2)^{2k}}, \qquad \rho = -\frac{x}{2},$$

$$\binom{x}{n} \sim \frac{(-)^n (n - x/2)^{-(x+1)}}{\Gamma(-x)} \Big[1 + \frac{(x)_3}{24(n - x/2)^2} + \frac{(x)_5(5x-2)}{5760(n - x/2)^4} + \frac{(x)_7(35x^2 - 42x + 16)}{29\,03040(n - x/2)^6} + \cdots\Big]. \tag{19}$$

It is clear that (14) is more powerful than (12). In illustration, suppose that we put $x = -\frac{1}{2}$ in (18) and (19). We get the respective equations

$$\pi = \frac{(n!)^4 2^{4n}}{n[(2n)!]^2}\Big[1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} + O(n^{-4})\Big]^2, \qquad n \to +\infty, \tag{20}$$

$$\pi = \frac{4(n!)^4 2^{4n}}{y[(2n)!]^2}\left[1 - \frac{1}{4y^2} + \frac{21}{32y^4} - \frac{671}{128y^6} + \frac{1\,80323}{2048y^8} - \frac{208\,98423}{8192y^{10}} + \frac{74263\,62705}{65536y^{12}} - \frac{187\,44094\,65055}{2\,62144y^{14}} + O(y^{-16})\right]^2, \tag{21}$$

$$y = 4n + 1, \qquad n \to +\infty$$

If $n = 20$, use of the first three terms in (20) gives π with an error of $0\,38 \quad 10^{-5}$ If $n = 10$, use of the first three terms in (21) gives π with an error of $-0\,69 \quad 10^{-8}$, while use of all the terms in (21) gives π with an error of $0\,56 \quad 10^{-16}$

Next we turn to a generalization of (12) Let

$$\begin{gathered} {}_ph_q(z) = \prod_{j=1}^{p} \Gamma(\alpha_j z + a_j) \Big/ \prod_{j=0}^{q} \Gamma(\beta_j z + b_j) \\ z \neq -(a_j + i)/\alpha_j, \quad j = 1, 2, \ldots, p, \quad i = 0, 1, 2, \ldots, \end{gathered} \tag{22}$$

and write

$$\begin{aligned} \alpha &= \tfrac{1}{2}(p - q) + \sum_{j=0}^{q} b_j - \sum_{j=1}^{p} a_j, & \beta &= \prod_{j=1}^{p} \alpha_j^{\alpha_j} \prod_{j=0}^{q} \beta_j^{-\beta_j}, \\ \mu &= \sum_{j=0}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j & \nu &= \prod_{j=1}^{p} \alpha_j^{\alpha_j - \frac{1}{2}} \prod_{j=0}^{q} \beta_j^{\frac{1}{2} - b_j} \end{aligned} \tag{23}$$

If $\mu > 0$, there exist computable constants h_j such that

$$\begin{gathered} {}_ph_q(z) = (2\pi)^{1/2(p-q)} \nu \beta^z \mu^{\mu z + \alpha} \left\{ \sum_{j=0}^{N-1} \frac{h_j}{\Gamma(\mu z + \alpha + j)} + O\left(\frac{1}{\Gamma(\mu z + \alpha + N)}\right) \right\}, \\ h_0 = 1, \quad z \neq -(\alpha + N + i)/\mu, \quad i = 0, 1, 2, \ldots, \\ |z| \to \infty, \quad |\arg z| \leqslant \pi - \epsilon, \quad \epsilon > 0 \end{gathered} \tag{24}$$

This may be deduced from (2) [see E M Wright (1935) and Braaksma (1963)] We also have for $\mu > 0$,

$$\begin{gathered} {}_pk_q(z) = \prod_{j=0}^{q} \Gamma(1 - b_j - \beta_j z) \Big/ \prod_{j=1}^{p} \Gamma(1 - a_j - \alpha_j z), \\ z \neq (1 - b_j + i)/\beta_j, \quad j = 0, 1, \ldots, q, \quad i = 0, 1, 2, \ldots, \\ {}_pk_q(z) = (2\pi)^{1/2(q-p)} \nu \beta^z \mu^{\mu z + \alpha} \\ \times \left\{ \sum_{j=0}^{N-1} (-)^j h_j \Gamma(1 - \mu z - \alpha - j) + O(\Gamma(1 - \mu z - \alpha - N)) \right\}, \\ z \neq (1 - \alpha - N + i)/\mu, \quad i = 0, 1, 2, \ldots, \\ |z| \to \infty, \quad |\arg(-z)| \leqslant \pi - \epsilon, \quad \epsilon > 0, \end{gathered} \tag{25}$$

$$
\begin{aligned}
{}_pk_q(-z) &= (2\pi)^{1/2(q-p)}{}_p\beta^{-z}\mu^{-\mu z+\alpha-\frac{1}{2}}\Gamma(\mu z+1-\alpha) \\
&\quad\times\left\{\sum_{j=0}^{N-1}\frac{h_j}{(-\mu z+\alpha)_j}+O\left(\frac{1}{(-\mu z+\alpha)_N}\right)\right\}, \\
&\qquad z\neq-(1-b_j+i)/\beta_j\,,\quad j=1,2,\ldots,q, \\
&\qquad z\neq(\alpha+N+i)/\mu,\qquad i=0,1,2,\ldots, \\
&\qquad |z|\to\infty,\qquad |\arg z|\leqslant\pi-\epsilon,\qquad \epsilon>0,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
[{}_ph_q(z)]^{-1} &= (2\pi)^{1/2(q-p)}{}_p\beta^{-1}\beta^{-z}\mu^{-\mu z+\frac{1}{2}-\alpha}\Gamma(\mu z+\alpha) \\
&\quad\times\left\{\sum_{j=0}^{N-1}\frac{d_j}{(-\mu z+1-\alpha)_j}+O\left(\frac{1}{(-\mu z+1-\alpha)_N}\right)\right\}, \\
&\qquad z\neq-(b_j+i)/\beta_j\,,\qquad j=0,1,\ldots,q, \\
&\qquad z\neq(1-\alpha+N+i)/\mu,\qquad i=0,1,2,\ldots, \\
&\qquad |z|\to\infty,\qquad |\arg z|\leqslant\pi-\epsilon,\qquad \epsilon>0.
\end{aligned}
\tag{27}
$$

Note that if in ${}_pk_q(-z)$, b_j and a_j are replaced by $1-b_j$ and $1-a_j$, respectively, then $[{}_ph_q(z)]^{-1}$ obtains.

An important special case occurs when each α_j and β_j is unity. In this event, with an obvious change in notation, let us write

$$
\begin{aligned}
&{}_pg_q(z)=\prod_{j=1}^{p}\Gamma(z+\alpha_j)\Big/\prod_{j=0}^{q}\Gamma(z+\rho_j), \\
&p\leqslant q,\quad \beta=q+1-p,\quad \varphi=(\beta-1)/2+\sum_{j=1}^{p}\alpha_j-\sum_{j=0}^{q}\rho_j\,.
\end{aligned}
\tag{28}
$$

Then there exist computable constants c_k,

$$
c_k\equiv c_k\left(p,q+1\,\middle|\,\begin{matrix}\alpha_p\\ \rho_0\,,\rho_q\end{matrix}\right),\qquad c_0=1,
\tag{29}
$$

such that

$$
\begin{aligned}
{}_pg_q(z) &= (2\pi)^{(1-\beta)/2}\beta^{\beta z-\varphi-\frac{1}{2}}\left\{\sum_{j=0}^{N-1}\frac{c_k}{\Gamma(\beta z-\varphi+j)}+O\left(\frac{1}{\Gamma(\beta z-\varphi+N)}\right)\right\}, \\
&\qquad z\neq-(\alpha_j+i),\quad j=1,2,\ldots,p, \\
&\qquad z\neq-(N+i-\varphi)/\beta,\quad i=0,1,2,\ldots,\quad |z|\to\infty, \\
&\qquad |\arg z|\leqslant\pi-\epsilon,\qquad \epsilon>0.
\end{aligned}
\tag{30}
$$

The c_k's can be generated by a recurrence formula, but this investigation is deferred to 5.11.5.

Chapter III HYPERGEOMETRIC FUNCTIONS

3 1 Elementary Hypergeometric Series

A vast body of special functions belong to the hypergeometric family or are related to functions of this class To introduce the subject, we record some well-known elementary expansions For the exponential function,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \qquad |z| < \infty, \tag{1}$$

and for the binomial function,

$$(1+z)^a = 1 + az + a(a-1)(z^2/2!) + \\ + a(a-1) \quad (a-k+1)(z^k/k!) + \quad , \qquad |z| < 1, \tag{2}$$

or with the aid of 2 1(7),

$$(1+z)^a = \sum_{k=0}^{\infty} \frac{(-a)_k(-z)^k}{k!}, \qquad |z| < 1 \tag{3}$$

Put $a = -1$ in (3) and integrate Then

$$z^{-1}\ln(1+z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k+1} = \sum_{k=0}^{\infty} \frac{(1)_k(1)_k(-z)^k}{(2)_k k!}, \qquad |z| < 1 \tag{4}$$

Again put $a = -1$ in (3), replace z by z^2 and integrate Thus,

$$z^{-1}\arctan z = \sum_{k=0}^{\infty} \frac{(1)_k(\frac{1}{2})_k(-z^2)^k}{(\frac{3}{2})_k k!} \qquad |z| < 1 \tag{5}$$

Multiply (1) by z^{a-1} and integrate Then

$$az^{-a}\int_0^z t^{a-1}e^t\,dt = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(a+1)_k k!}, \qquad R(a) > 0 \quad |z| < \infty \tag{6}$$

In

$$\cos z = \sum_{k=0}^{\infty} \frac{(-)^k z^{2k}}{(2k)!}, \qquad |z| < \infty, \tag{7}$$

use 2.1(10) or 2.3(2) to get

$$\cos z = \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{(1/2)_k k!}, \qquad |z| < \infty. \tag{8}$$

Similarly,

$$z^{-1} \sin z = \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{(3/2)_k k!}, \qquad |z| < \infty. \tag{9}$$

The Bessel function of the first kind of order ν can be defined by the expansion

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{(\nu + 1)_k k!}, \qquad |z| < \infty, \tag{10}$$

and we immediately see that

$$\cos z = (\pi z/2)^{1/2} J_{-1/2}(z), \qquad \sin z = (\pi z/2)^{1/2} J_{1/2}(z). \tag{11}$$

In each of the above expansions, the general term is of the form

$$(u)_k (v)_k x^k / (w)_k k! \tag{12}$$

provided we understand that if as in $\cos z$, $(u)_k$ and $(v)_k$ are not present, then these quantities are said to be empty and are treated as unity. Thus for $\cos z$, $x = -z^2/4$ and $w = \frac{1}{2}$. Likewise for $z^{-1} \arctan z$, $x = -z^2$, $u = 1$, $v = \frac{1}{2}$, $w = \frac{3}{2}$. In (12), x is called the variable. Also u and v are called numerator parameters while w is a denominator parameter.

The preceding discussion suggests that we consider the form

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} [(a)_k (b)_k z^k / (c)_k k!]. \tag{13}$$

Often, as a space saver, we write

$${}_2F_1(a, b; c; z) = {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right), \tag{14}$$

and where no confusion can arise, we simply refer to the latter as a ${}_2F_1$ Some authors drop the subscripts attached to F However, we shall not follow this practice In illustration,

$$z^{-1}\ln(1+z) = {}_2F_1(1, 1, 2, -z), \tag{15}$$

$$z^{-1}\arctan z = {}_2F_1(1, \tfrac{1}{2}, \tfrac{3}{2}, -z^2) \tag{16}$$

Affixation of the subscripts to F is useful, for it is easy to have a notation for sums of terms like (12) where only one or no numerator term is present or only one or no denominator term is present Thus,

$${}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\middle| z\right) = {}_1F_1(a, c, z) = \sum_{k=0}^{\infty} [(a)_k z^k/(c)_k k!], \tag{17}$$

and so

$$az^{-a}\int_0^z t^{a-1}e^t\,dt = {}_1F_1(a, a+1, z) \tag{18}$$

Similarly,

$$\Gamma(\nu+1)(z/2)^{-\nu}J_\nu(z) = {}_0F_1(\nu+1, -z^2/4), \tag{19}$$

$$e^z = {}_0F_0(z) = {}_1F_1(a, a, z) \tag{20}$$

In the latter formula as well as in (13), if a numerator and denominator parameter coalesce, the parameter can be omitted and the subscripts attached to F are each reduced by unity Thus,

$$(1+z)^a = {}_2F_1(-a, b, b, -z) = {}_1F_0(-a, -z) \tag{21}$$

Clearly (13) is symmetric in a and b If $a = -m$, m a positive integer or zero, then (13) is a polynomial provided c is not a negative integer or zero Thus

$${}_2F_1(-m, b, c, z) = \sum_{k=0}^{m} [(-m)_k (b)_k z^k/(c)_k k!] \tag{22}$$

An alternative representation for (22) follows by turning the series around Thus,

$${}_2F_1(-m, b, c, z) = \frac{(b)_m(-)^m z^m}{(c)_m}\,{}_2F_1(-m, 1-m-c, 1-m-b, z^{-1}),$$

$$c \quad \text{not a negative integer or zero} \tag{23}$$

If c is a negative integer or zero, say $c = -n$, and if for convenience to

simplify the discussion we suppose that b is not a negative integer or zero, then

$$ {}_2F_1(-m, b; -n; z) \quad \text{is not defined if} \quad n < m, \tag{24} $$

$$ {}_2F_1(-m, b; -n; z) = {}_1F_0(b; z) = (1 - z)^{-b} \quad \text{if} \quad n = m, \tag{25} $$

$$ \begin{aligned} {}_2F_1(-m, b; -n; z) &= \sum_{k=0}^{m} \frac{(-m)_k (b)_k z^k}{(-n)_k k!} + \sum_{k=n+1}^{\infty} \frac{(-m)_k (b)_k z^k}{(-n)_k k!} \\ &= \frac{(n-m)!\,(b)_m z^m}{n!} {}_2F_1\left(\begin{matrix} -m, n+1-m \\ 1-m-b \end{matrix} \middle| z^{-1}\right) \\ &\quad + \frac{(n-m)!\, m!(b)_{n+1}(-)^{m+n} z^{n+1}}{n!(n+1)!} \\ &\quad \times {}_2F_1\left(\begin{matrix} n+1-m, n+1+b \\ n+2 \end{matrix} \middle| z\right), \quad \text{if } n > m. \end{aligned} \tag{26} $$

3.2. A Generalization of the ${}_2F_1$

Instead of discussing the ${}_2F_1$, ${}_1F_1$, etc., we might just as well study a generalized hypergeometric series with an arbitrary number of numerator and denominator parameters. We therefore consider

$$ \begin{aligned} {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \rho_1, \rho_2, \ldots, \rho_q; z) &= {}_pF_q\left(\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \rho_1, \rho_2, \ldots, \rho_q \end{matrix} \middle| z\right) \\ &= \sum_{k=0}^{\infty} \left[\prod_{h=1}^{p} (\alpha_h)_k z^k \Big/ \prod_{h=1}^{q} (\rho_h)_k k!\right]. \end{aligned} \tag{1} $$

The latter is formal in the sense that for the present we do not consider the convergence of the series. This aspect is taken up in 3.3. Where no confusion can arise, we simply refer to (1) as a ${}_pF_q$. It is often convenient to employ a contracted notation and write (1) in the abbreviated form

$$ {}_pF_q(\alpha_p; \rho_q; z) = {}_pF_q\left(\begin{matrix} \alpha_p \\ \rho_q \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} [(\alpha_p)_k z^k / (\rho_q)_k k!]. \tag{2} $$

Thus $\Gamma(\alpha_p + k)$ is interpreted as $\prod_{j=1}^{p} \Gamma(\alpha_j + k)$; $(\alpha_p)_k$ as $\prod_{j=1}^{p} (\alpha_j)_k$; α_p as $\prod_{j=1}^{p} \alpha_j$, etc. An empty term is treated as unity so that, for example, if $p = 2$, $(\alpha_h)_k = 1$ for $h > 2$. The α_h's and ρ_h's are called numerator and denominator parameters, respectively, and z is called the variable. The ${}_pF_q$ is symmetric in its numerator parameters, and likewise in its denominator parameters. If a numerator parameter and a denominator

parameter coalesce, then omit the parameter, whence the ${}_pF_q$ becomes a ${}_{p-1}F_{q-1}$ The ${}_pF_q$ series terminates and, therefore, is a polynomial if a numerator parameter is a negative integer or zero, provided that no denominator parameter is a negative integer or zero The results 3 1(23–26) are easily generalized

$$ {}_{p+1}F_{q+1}\left(\begin{matrix}-m & \alpha_p\\ c & \rho_q\end{matrix}\middle| z\right) = \frac{(\alpha_p)_m(-)^m z^m}{(c)_m(\rho_q)_m} \times {}_{q+2}F_p\left(\begin{matrix}-m, 1-m-c & 1-m-\rho_q\\ & 1-m-\alpha_p\end{matrix}\middle| \frac{(-)^{p+q+1}}{z}\right) \tag{3} $$

where neither c nor any ρ_h is a negative integer or zero If no ρ_h is a negative integer or zero, if, for convenience, to simplify the discussion, no α_h is a negative integer or zero, and if c is a negative integer or zero, $c = -n$ then

$$ {}_{p+1}F_{q+1}\left(\begin{matrix}-m & \alpha_p\\ n & \rho_q\end{matrix}\middle| z\right) \quad \text{is not defined if} \quad n < m, \tag{4} $$

$$ {}_{p+1}F_{q+1}\left(\begin{matrix}-m & \alpha_p\\ -n & \rho_q\end{matrix}\middle| z\right) = {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) \quad \text{if} \quad n = m \tag{5} $$

$$ \begin{aligned} {}_{p+1}F_{q+1}\left(\begin{matrix}-m & \alpha_p\\ -n & \rho_q\end{matrix}\middle| z\right) &= \frac{(n-m)!\,(\alpha_p)_m z^m}{n!(\rho_q)_m} \\ &\quad\times {}_{q+2}F_p\left(\begin{matrix}-m\ n+1-m & 1-m-\rho_q\\ & 1-m-\alpha_p\end{matrix}\middle| \frac{(-)^{p+q+1}}{z}\right) \\ &\quad + \frac{(n-m)!\,m!(\alpha_p)_{n+1}(-)^{m+n}z^{n+1}}{n!(n+1)!\,(\rho_q)_{n+1}} \\ &\quad\times {}_{p+1}F_{q+1}\left(\begin{matrix}n+1-m\ n+1 & +\alpha_p\\ n+2\ n+1 & +\rho_q\end{matrix}\middle| z\right) \quad \text{if } n > m \end{aligned} \tag{6} $$

If in (3), we let $c = -m + \epsilon$ compose the finite sums as indicated, and then let $\epsilon \to 0$, we see that the truncated hypergeometric series (1) may be expressed as a hypergeometric polynomial That is,

$$ y_m\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) = \sum_{k=0}^{m} \frac{(\alpha_p)_k z^k}{(\rho_q)_k k!} = \frac{(\alpha_p)_m z^m}{(\rho_q)_m m!}\, {}_{q+2}F_p\left(\begin{matrix}-m & 1-m-\rho_q & 1\\ & 1-m-\alpha_p & \end{matrix}\middle| \frac{(-)^{p+q+1}}{z}\right) \tag{7} $$

where no ρ_h is a negative integer or zero

This chapter for the most part deals with the ${}_2F_1$ The ${}_1F_1$ and ${}_0F_1$ cases are the subjects of Chapter IV, while the ${}_pF_q$ and a generalization of the ${}_pF_q$ are studied in Chapter V However, in some instances where theorems for a general ${}_pF_q$ differ essentially from those for a ${}_2F_1$ by notation only, then we state results for the general case in this chapter

It calls for us to remark that the literature is vast with information on the ${}_2F_1$, ${}_1F_1$, and ${}_0F_1$ cases. Much less can be said for the general ${}_pF_q$, and for this reason one studies the former cases independently of the latter.

We have already noted in 3.1 some elementary functions which can be expressed by the ${}_2F_1$ symbol. Other important cases of this family are Legendre functions (see 6.2.3) and the classical orthogonal polynomials (see Chapter VIII). For ease in the applications, we have prepared a table of the most common functions of mathematical physics which may be characterized by the ${}_pF_q$ symbol. This is given in Chapter VI.

Some general sources for material on the ${}_pF_q$ and its special cases are Erdélyi *et al.* (1953), Kratzer and Franz (1960), Kuznecov (1965), Lebedev (1965), Luke (1962a), MacRobert (1962a), Mitrinović and Djoković (1964), Poole (1960), Rainville (1960), Schäfke (1963), Slater (1966), Sneddon (1956), Snow (1952), Watson (1945), and Whittaker and Watson (1927). See also the references given in 4.1, 6.2.3, 6.2.6, 8.1, and the references quoted in the handbook edited by Abramowitz and Stegun (1964). Other useful handbooks are Byrd and Friedman (1954), Erdélyi *et al.* (1954). Gröbner and Hofreiter (1949, 1950), Magnus and Oberhettinger (1948, 1954, 1966), Mangulis (1965), Oberhettinger (1957), and Ryshik and Gradstein (1957, 1965). For descriptions of numerical mathematical tables embracing the entire spectrum of transcendental functions, see Fletcher *et al.* (1962), Lebedev and Feodorova (1956), and Burunova (1959). See also the handbook edited by Abramowitz and Stegun (1964) cited above.

3.3. Convergence of the ${}_pF_q$ Series

We suppose that none of the numerator or denominator parameters of 3.2(1) is a negative integer or zero. Let u_k be the coefficient of z^k in 3.2(1). Then

$$\frac{u_{k+1}z^{k+1}}{u_k z^k} = \frac{z(k+\alpha_p)}{(k+\rho_q)(k+1)} = zk^{p-q-1}\left\{1 + \frac{\eta - 1}{k} + O(k^{-2})\right\},$$

$$\eta = \sum_{h=1}^{p} \alpha_h - \sum_{h=1}^{q} \rho_h . \tag{1}$$

Application of the ratio test shows that the series

$$\begin{aligned} &\text{converges for all finite } z \text{ if } p \leqslant q, \\ &\text{converges for } |z| < 1 \text{ if } p = q+1, \\ &\text{diverges for all } z,\ z \neq 0 \text{ if } p > q+1. \end{aligned} \tag{2}$$

Following Bromwich (1949, pp 41, 241) or Knopp (1949, p 401), we can show that the ${}_{q+1}F_q$ series is

$$\begin{aligned} &\text{absolutely convergent for} \quad |z| = 1 \quad \text{if} \quad R(\eta) < 0, \\ &\text{conditionally convergent for} \quad |z| = 1, \quad z \neq 1 \quad \text{if} \quad 0 \leqslant R(\eta) < 1, \\ &\text{divergent for} \quad |z| = 1 \quad \text{if} \quad 1 \leqslant R(\eta) \end{aligned} \tag{3}$$

3 4 Elementary Relations

We infer from the examples considered in 3 1 that derivatives of hypergeometric series are series of the same kind The following relations are of this type

$$\frac{d^n}{dz^n}\, {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) = \frac{(\alpha_p)_n}{(\rho_q)_n}\, {}_pF_q\left(\begin{matrix}\alpha_p + n\\ \rho_q + n\end{matrix}\middle| z\right) \tag{1}$$

$$\frac{d^n}{dz^n}\left[z^\delta\, {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right)\right] = (\delta - n + 1)_n z^{\delta - n}\, {}_{p+1}F_{q+1}\left(\begin{matrix}\delta + 1, \alpha_p\\ \delta + 1 - n, \rho_q\end{matrix}\middle| z\right) \tag{2}$$

If $(\delta + 1 - n)$ is a negative integer or zero, a more convenient form of *(2)* is

$$\frac{d^n}{dz^n}\left[z^\delta\, {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right)\right]$$
$$= \frac{(\alpha_p)_n\, \delta n!}{(\rho_q)_{n-\delta}(n-\delta)!}\, {}_{p+1}F_{q+1}\left(\begin{matrix}\alpha_p + n - \delta, n + 1\\ \rho_q + n - \delta, n + 1 - \delta\end{matrix}\middle| z\right) \tag{3}$$

$$\frac{d^n}{dz^n}\left[z^{\sigma+n-1}\, {}_{p+1}F_q\left(\begin{matrix}\sigma, \alpha_p\\ \rho_q\end{matrix}\middle| z\right)\right]$$
$$= (\sigma)_n z^{\sigma-1}\, {}_{p+1}F_q\left(\begin{matrix}\sigma + n, \alpha_p\\ \rho_q\end{matrix}\middle| z\right) \tag{4}$$

$$\frac{d^n}{dz^n}\left[z^{\sigma-1}\, {}_pF_{q+1}\left(\begin{matrix}\alpha_p\\ \sigma, \rho_q\end{matrix}\middle| z\right)\right]$$
$$= (\sigma - n)_n z^{\sigma-1-n}\, {}_pF_{q+1}\left(\begin{matrix}\alpha_p\\ \sigma - n, \rho_q\end{matrix}\middle| z\right) \tag{5}$$

$$\frac{d^n}{dz^n}\left[z^\delta(1-z)^{a+b-c}\, {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle| z\right)\right]$$
$$= (\delta - n + 1)_n z^{\delta-n}\, {}_3F_2\left(\begin{matrix}c - a, c - b, \delta + 1\\ c, \delta + 1 - n\end{matrix}\middle| z\right) \tag{6}$$

$$\frac{d^n}{dz^n}\left[z^{n+c-1}(1-z)^{n+a+b-c}\, {}_2F_1\left(\begin{matrix}a + n, b + n\\ c + n\end{matrix}\middle| z\right)\right]$$
$$= (c)_n z^{c-1}(1-z)^{a+b-c}\, {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle| z\right) \tag{7}$$

$$\frac{d^n}{dz^n}\left[z^{c-a+n-1}(1-z)^{a+b-c}\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle|\, z\right)\right]$$
$$= (c-a)_n z^{c-a-1}(1-z)^{a+b-c-n}\,{}_2F_1\left(\begin{matrix}a-n, b\\ c\end{matrix}\middle|\, z\right). \tag{8}$$

$$\frac{d^n}{dz^n}\left[z^{c-1}(1-z)^{a+b-c}\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle|\, z\right)\right]$$
$$= (c-n)_n z^{c-n-1}(1-z)^{a+b-c-n}\,{}_2F_1\left(\begin{matrix}a-n, b-n\\ c-n\end{matrix}\middle|\, z\right). \tag{9}$$

$$\frac{d^n}{dz^n}\left[(1-z)^{a+b-c}\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle|\, z\right)\right]$$
$$= \frac{(c-a)_n(c-b)_n}{(c)_n}(1-z)^{a+b-c-n}\,{}_2F_1\left(\begin{matrix}a, b\\ c+n\end{matrix}\middle|\, z\right). \tag{10}$$

$$\frac{d^n}{dz^n}\left[z^{\alpha}(1-z)^{\beta}\right]$$
$$= (\alpha-n+1)_n z^{\alpha-n}(1-z)^{\beta-n}\,{}_2F_1\left(\begin{matrix}-n, \alpha+1+\beta-n\\ \alpha+1-n\end{matrix}\middle|\, z\right),$$
$$\alpha \text{ not an integer} < n, \tag{11}$$

$$\frac{d^n}{dz^n}\left[z^{s}(1-z)^{\beta}\right]$$
$$= \frac{n!(-\beta)_{n-s}}{(n-s)!}(1-z)^{\beta-n}\,{}_2F_1\left(\begin{matrix}-s, \beta+1\\ n-s+1\end{matrix}\middle|\, z\right),$$
$$s \text{ an integer}, \quad s \leqslant n. \tag{12}$$

$$\frac{d^n}{dz^n}\left[z^{c-1}(1-z)^{b-c+n}\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle|\, z\right)\right]$$
$$= (c-n)_n z^{c-1-n}(1-z)^{b-c}\,{}_2F_1\left(\begin{matrix}a-n, b\\ c-n\end{matrix}\middle|\, z\right). \tag{13}$$

$$\frac{d^n}{dz^n}\left[(1-z)^{a+n-1}\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle|\, z\right)\right]$$
$$= \frac{(-)^n(a)_n(c-b)_n}{(c)_n}(1-z)^{a-1}\,{}_2F_1\left(\begin{matrix}a+n, b\\ c+n\end{matrix}\middle|\, z\right). \tag{14}$$

$$A(z) = {}_2F_1\left(\begin{matrix}\nu+1, \nu+\mu+1\\ \nu+\mu+\lambda+2\end{matrix}\middle|\, z\right)$$
$$= \frac{(-)^{\mu+1}(\nu+\mu+\lambda+1)!}{\lambda!\,\nu!(\nu+\mu)!\,(\mu+\lambda)!}\frac{d^{\nu+\mu}}{dz^{\nu+\mu}}\left[(1-z)^{\mu+\lambda}\frac{d^{\lambda}}{dz^{\lambda}}\{z^{-1}\ln(1-z)\}\right], \tag{15}$$

where ν, μ, λ are positive integers or zero. Note that the right-hand side of (15) gives the analytic continuation of the ${}_2F_1$ on the left throughout the entire complex plane, the positive real axis from 1 to ∞ excluded. Equation (15) is readily proved from (1) and (10) with the aid of 3.1(15).

Equations (1)–(5) follow by differentiation To prove (6), first transform the ${}_2F_1$ on the left using 3 8(2) and then apply (1) Equations (7)–(12) are special cases of (6) The proofs for (13), (14) are not so direct We derive these equations and as a by-product get alternative proofs of (9), (10)

It is convenient to set

$$A = \frac{d^n}{dz^n}\left\{z^\alpha(1-z)^\beta\, {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\middle| z\right)\right\}$$

We first assume that α is not an integer less than n Use (11) with α replaced by $\alpha + k$ Then

$$A = z^{\alpha-n}(1-z)^{\beta-n}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}(\alpha+k+1-n)_n z^k$$

$$\times\sum_{m=0}^{n}\frac{(-n)_m(\alpha+k+1-n+\beta)_m}{(\alpha+k+1-n)_m m!}z^m,$$

$$A = z^{\alpha-n}(1-z)^{\beta-n}(\alpha+1-n)_n\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(\alpha+1)_k z^k}{(c)_k(\alpha+1-n)_k k!} \tag{16}$$

$$\times\, {}_4F_3\left(\begin{matrix}-k, -n, 1-c-k, n-\alpha-\beta-k\\ 1-a-k, 1-b-k, -\alpha-k\end{matrix}\middle| 1\right)$$

If $\alpha = c - 1$ and $\beta = n + b - c$, the ${}_4F_3$ becomes a ${}_2F_1$ which is readily summed using 3 13 1(1), and A reduces to (13) If $\alpha = c - 1$ and $\beta = a + b - c$, the ${}_4F_3$ becomes a Saalschützian ${}_3F_2$ which can be summed by 3 13 3(2) This leads to the proof of (9)

Suppose now that α is an integer, say s, and $s \leqslant n$ We could get an expansion for A like (16) starting from (12) However, the desired expansion can be obtained from (16) if there $\sum_{k=0}^{\infty}$ is replaced by $\sum_{k=n-s}^{\infty}$ We have

$$A = (1-z)^{\beta-n}\sum_{k=0}^{\infty}\frac{(a)_{k+n-s}(b)_{k+n-s}\Gamma(1+k+n)z^k}{(c)_{k+n-s}\Gamma(1+k+n-s)k!}$$

$$\times\, {}_4F_3\left(\begin{matrix}-k-n+s, -n, 1-c-k-n+s, -\beta-k\\ 1-a-k-n+s, 1-b-k-n+s, -k-n\end{matrix}\middle| 1\right), \tag{17}$$

where the ${}_4F_3$ is to be interpreted as the sum of the $(n + k + 1 - s)$ or $(n + 1)$ terms whichever is least If $s = 0$ and $\beta = a + n - 1$, the ${}_4F_3$ becomes a ${}_2F_1$ Sum this using 3 13 1(1) whence A gives (14) If $s = 0$ and $\beta = a + b - c$, the ${}_4F_3$ reduces to a ${}_3F_2$ which is Saalschützian Use 3 13 3(2) to sum the ${}_3F_2$, and A reduces to (10)

The six functions

$$ {}_2F_1(a \pm 1, b; c; z), \qquad {}_2F_1(a, b \pm 1, c; z), \qquad {}_2F_1(a, b; c \pm 1; z) \tag{18} $$

are called *contiguous* to ${}_2F_1(a, b; c; z)$. We use the notation ${}_2F_1(a+)$ and ${}_2F_1(a-)$ to designate the ${}_2F_1$ with a replaced by $(a + 1)$ and $(a - 1)$, respectively, etc. Gauss proved that between the ${}_2F_1$ and any two functions contiguous to it, there exists a linear relation with coefficients which are linear in z. There are fifteen relations of this kind. Only four of the fifteen are really independent, as all others may be obtained by elimination and use of the fact that the ${}_2F_1$ is symmetric in a and b. In view of symmetry, it is sufficient to record only nine of the fifteen relations. They are as follows:

$$ (c - a)\,{}_2F_1(a-) + (2a - c - az + bz)\,{}_2F_1 + a(z - 1)\,{}_2F_1(a+) = 0. \tag{19} $$

$$ \begin{aligned} &c(c - 1)(z - 1)\,{}_2F_1(c-) \\ &\quad + c[c - 1 - (2c - a - b - 1)z]\,{}_2F_1 + (c - a)(c - b)z\,{}_2F_1(c+) = 0. \end{aligned} \tag{20} $$

$$ c[a + (b - c)z]\,{}_2F_1 - ac(1 - z)\,{}_2F_1(a+) + (c - a)(c - b)z\,F(c+) = 0. \tag{21} $$

$$ c(1 - z)\,{}_2F_1 - c\,{}_2F_1(a-) + (c - b)z\,{}_2F_1(c+) = 0. \tag{22} $$

$$ (b - a)\,{}_2F_1 + a\,{}_2F_1(a+) - b\,{}_2F_1(b+) = 0. \tag{23} $$

$$ (c - a - b)\,{}_2F_1 + a(1 - z)\,{}_2F_1(a+) - (c - b)\,F(b-) = 0. \tag{24} $$

$$ (c - a - 1)\,{}_2F_1 + a\,{}_2F_1(a+) - (c - 1)\,{}_2F_1(c-) = 0. \tag{25} $$

$$ (b - a)(1 - z)\,{}_2F_1 - (c - a)\,{}_2F_1(a-) + (c - b)\,{}_2F_1(b-) = 0. \tag{26} $$

$$ \begin{aligned} &[a - 1 + (b + 1 - c)z]\,{}_2F_1 \\ &\quad + (c - a)\,{}_2F_1(a-) - (c - 1)(1 - z)\,{}_2F_1(c-) = 0. \end{aligned} \tag{27} $$

These are readily verified by expanding in power series and showing that the coefficients of all powers of z vanish.

If m, n, and s are integers, then ${}_2F_1(a + m, b + n; c + s; z)$ can be expressed by repeated application of the contiguous relations as a linear combination of ${}_2F_1$ and one of its contiguous functions with coefficients which are rational functions of a, b, c, and z.

Formulas (1)–(5) give rise to difference-differential equations. Many such properties for the ${}_2F_1$ follow upon combining most of the results in Eqs. (1)–(10) with (19)–(27).

We can invoke the confluence principle [see 3.5(6)] to get recursion relations for the ${}_1F_1$. Thus in (19), replace z by z/b and let $b \to \infty$.

Then

$$(c - a)\,{}_1F_1(a-) + (2a - c + z)\,{}_1F_1 - a\,{}_1F_1(a+) = 0 \tag{28}$$

Similarly,

$$c(c - 1)\,{}_1F_1(c-) + c(1 - c - z)\,{}_1F_1 + (c - a)z\,{}_1F_1(c+) = 0, \tag{29}$$

and from this we deduce

$$c(c - 1)[-{}_0F_1(c-) + {}_0F_1] + z\,{}_0F_1(c+) = 0 \tag{30}$$

In particular, see 3 1(19)

$$J_{\nu+1}(z) + J_{\nu-1}(z) = (2\nu/z)J_\nu(z), \tag{31}$$

and from (1),

$$zJ_\nu(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z) \tag{32}$$

Contiguous and difference-differential properties for the ${}_3F_2$ and ${}_4F_3$ have been given by Bailey (1954) Results of this type for the ${}_pF_q$ have been studied by Rainville (1960, pp 80–85) Recursion and difference differential formulas for a form simply related to ${}_2F_1(-n, n + \lambda, \beta + 1, z)$ are given in 8 2 A recursion formula for a generalization of the latter is given in 12 2 See Sections 12 3 12 4 for the development of recursion systems for other types of hypergeometric functions

3.5 The Confluence Principle

With z bounded,

$$\begin{aligned}\lim_{\sigma\to\infty}(1 - z/\sigma)^{-\sigma} &= \lim_{\sigma\to\infty}\exp[-\sigma\ln(1 - z/\sigma)]\\ &= e^z \lim_{\sigma\to\infty}\exp\{(z^2/2\sigma)[1 + O(z/\sigma)]\} = e^z,\end{aligned} \tag{1}$$

and in our hypergeometric notation

$$\lim_{\sigma\to\infty}{}_1F_0(\sigma\ z/\sigma) = {}_0F_0(z) = e^z \qquad z \text{ bounded} \tag{2}$$

We can get this result from another point of view Now the general term in the expansion of $(1 - z/\sigma)^{-\sigma}$ is

$$v_k z^k/k! = \Gamma(\sigma + k)z^k/\Gamma(\sigma)\sigma^k k! \tag{3}$$

and from 2 11(11), with k fixed,

$$\lim_{|\sigma|\to\infty} v_k = 1 \tag{4}$$

Thus with z fixed,

$$\lim_{|\sigma|\to\infty}(1-z/\sigma)^{-\sigma} = \lim_{|\sigma|\to\infty}\sum_{k=0}^{\infty}(v_k z^k/k!) = \sum_{k=0}^{\infty}(z^k/k!)\lim_{|\sigma|\to\infty} v_k$$

$$= \sum_{k=0}^{\infty} z^k/k! = e^z. \tag{5}$$

Again, if in ${}_2F_1(a, b; c; z)$, z is replaced by z/b, we get a power series in z whose radius of convergence is $|b|$. The latter defines an analytic function with singularities at $z = 0$, b, and ∞ [see 3.7(1)]. Let $|b| \to \infty$. The limiting form defines an entire function with a singularity at $z = \infty$ which is a confluence of the two singularities b and ∞ of ${}_2F_1(a, b; c; z/b)$. Thus,

$$\lim_{|b|\to\infty} {}_2F_1(a, b; c; z/b) = {}_1F_1(a; c; z) \tag{6}$$

and the ${}_1F_1$ is a confluent form of the ${}_2F_1$. For this reason, the ${}_1F_1$ is called a confluent hypergeometric function. A rather thorough study of the ${}_1F_1$ is given in Chapter IV. A natural generalization of (6) is the statement

$$\lim_{|\sigma|\to\infty} {}_{p+1}F_q\left(\begin{matrix}\alpha_p, \sigma\\ \rho_q\end{matrix}\middle| z/\sigma\right) = {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right), \qquad q \geqslant p. \tag{7}$$

This limit process is called a confluence with respect to σ and the resulting limit of such a process is called a confluent limit. The importance of the result (7) lies in the fact that known representations for ${}_{p+1}F_q$ may be used to deduce similar type representations for ${}_pF_q$ which do not follow from the former by an obvious suppression of a numerator parameter.

In (7), the confluence is with respect to a numerator parameter of the ${}_{p+1}F_q$. However, the limit process can be invoked with respect to a denominator parameter, and it can be proved that

$$\begin{aligned}
&\lim_{|\beta|\to\infty} {}_pF_{q+1}\left(\begin{matrix}\alpha_p\\ \rho_q, \beta\end{matrix}\middle| \beta z\right) = {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right), \qquad p \leqslant q+1,\\
&|\arg\beta| \leqslant \pi - \delta, \qquad 0 < \delta \leqslant \pi/2,\\
&|z| < R \qquad \text{if} \quad |\arg\beta| \leqslant \pi/2,\\
&|z| < |\sin(\arg\beta)|\, R \qquad \text{if} \quad \pi/2 \leqslant |\arg\beta| \leqslant \pi - \delta,
\end{aligned} \tag{8}$$

where $R = 1$ if $p = q + 1$ and $R = \infty$ if $p \leqslant q$.

The results (7), (8) are special cases of some general theorems proved by Fields (1966) We next present a summary of his findings

Theorem 1. *Let*

$$\sum_{k=0}^{\infty} a_k z^k < \infty \qquad |z| < R \tag{9}$$

Then

$$F(z, \sigma) = \sum_{k=0}^{\infty} [a_k(\sigma)_k/k!](z/\sigma)^k, \qquad |z/\sigma| < R, \tag{10}$$

and $F(z, \sigma)$ can be rearranged to read

$$F(z, \sigma) = \sum_{j=0}^{\infty} g_j(z)\sigma^{-j}, \qquad |z/\sigma| < R \tag{11}$$

That is, $F(z, \sigma)$ converges for $|z/\sigma| < R$, where the $g_j(z)$ are entire functions of z given explicitly by (16) *below Further, for $j \geqslant 1$, $g_j(z)$ can be expressed in terms of the derivatives of $g_0(z) = g(z)$*

PROOF From the ratio test and (9), it follows that $F(z, \sigma)$ converges for $|z| < |\sigma|R$ For $\sigma \neq 0$, k an integer $\geqslant 0$,

$$(\sigma)_k/\sigma^k = (1 + k/\sigma)^{-1} \prod_{j=0}^{k} (1 + j/\sigma) = \sum_{j=0}^{k} \alpha_{j,k}\sigma^{-j}, \qquad \alpha_{j,k} > 0 \tag{12}$$

Clearly $F(z, \sigma)$ is majorized by the convergent series

$$\sum_{k=0}^{\infty} \frac{|a_k||z|^k}{k!} \sum_{j=0}^{k} \alpha_{j,k}|\sigma|^{-j} = \sum_{k=0}^{\infty} \frac{|a_k|(|\sigma|)_k}{k!} |z/\sigma|^k, \qquad |z/\sigma| < R \tag{13}$$

Thus $F(z, \sigma)$ can be rearranged in descending powers of σ and since σ is arbitrary, the $g_j(z)$ converge for arbitrary z We next identify the $\alpha_{j,k}$ in (12) From 2 11(12), with $z = \sigma$, $a = k$, and $b = 0$, we have with the aid of 2 8(6, 11),

$$\alpha_{j,k} = [(1 - k)_j/j!]B_j^{(k)} \tag{14}$$

Note that

$$\begin{aligned} \alpha_{j,k} &= 1 \quad \text{if} \quad j = 0, \quad k \geqslant 0 \\ &= 0 \quad \text{if} \quad j \geqslant 1 \quad k = 0 \quad \text{or} \quad k = 1 \end{aligned} \tag{15}$$

It follows that

$$g_j(z) = \sum_{k=0}^{\infty} \frac{a_k\alpha_{j,k}z^k}{k!} = \sum_{k=0}^{\infty} \frac{a_k(1 - k)_j B_j^{(k)} z^k}{j!\,k!} \tag{16}$$

Now for $k, j \geqslant 1$, $k^{-1}B_j^{(k)}$ is a polynomial in k of order $(j-1)$. If Δ denotes the forward difference operator with respect to k, and m is an integer $\geqslant 0$, the relation

$$\Delta^m(1+j-k)_r = (-r)_m(1+j-k)_{r-m}$$

implies that for $j \geqslant 1$,

$$B_j^{(k)} = k\sum_{r=0}^{j-1} \frac{(-)^r(1+j-k)_r}{r!}\left[\Delta^r\left(\frac{B_j^{(k)}}{k}\right)\right]_{k=j+1}. \tag{17}$$

Combining (16) and (17), we see that for $j \geqslant 1$,

$$g_j(z) = \sum_{r=0}^{j-1} \frac{z^{1+j+r}}{j!\,r!} g_0^{(1+j+r)}(z)\left[\Delta^r\left(\frac{B_j^{(k)}}{k}\right)\right]_{k=j+1}, \tag{18}$$

and this completes the proof of Theorem 1.

Since the coefficients of $z^{1+j+r}g_0^{(1+j+r)}(z)$ in (18) are independent of $g_0(z)$, the coefficients can be deduced from the special case

$$F(z,\sigma) = (1-z/\sigma)^{-\sigma} = e^z \exp\left\{\sum_{j=2}^{\infty}(\sigma/j)(z/\sigma)^j\right\}, \qquad g_0(z) = e^z. \tag{19}$$

The first few $g_j(z)$ are as follows:

$$\begin{aligned}
&g_0(z) = g(z) = \sum_{k=0}^{\infty} a_k z^k/k!, \qquad g_1(z) = (z^2/2)g^{(2)}(z),\\
&g_2(z) = (z^3/3)g^{(3)}(z) + (z^4/8)g^{(4)}(z),\\
&g_3(z) = (z^4/4)g^{(4)}(z) + (z^5/6)g^{(5)}(z) + (z^6/48)g^{(6)}(z),\\
&g_4(z) = (z^5/5)g^{(5)}(z) + (13z^6/72)g^{(6)}(z) + (z^7/24)g^{(7)}(z) + (z^8/384)g^{(8)}(z).
\end{aligned} \tag{20}$$

By virtue of Theorem 1 and 3.4(1), we have

$$\begin{aligned}
{}_{p+1}F_q\left(\begin{matrix}\alpha_p, \sigma\\ \rho_q\end{matrix}\middle| z/\sigma\right) &= {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) + \frac{A_2(z)}{2\sigma} + \frac{1}{24\sigma^2}(8A_3(z) + 3A_4(z))\\
&\quad + \frac{1}{48\sigma^3}(12A_4(z) + 8A_5(z) + A_6(z)) + \frac{1}{5760\sigma^4}\\
&\quad \times (1152A_5(z) + 1040A_6(z) + 240A_7(z) + 15A_8(z)) + \cdots,\\
A_r(z) &= z^r\frac{d^r}{dz^r}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) = \frac{(\alpha_p)_r z^r}{(\rho_q)_r}\,{}_pF_q\left(\begin{matrix}\alpha_p + r\\ \rho_q + r\end{matrix}\middle| z\right),\\
&\quad p = q, \quad |z| < |\sigma|; \qquad p < q, \quad |z| < \infty.
\end{aligned} \tag{21}$$

Another kind of confluence is suggested by the classical relationship connecting the Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ [see 8 2(2)] and the Bessel function $J_\alpha(z)$,

$$\lim_{n\to\infty} n^{-\alpha}P_n^{(\alpha,\beta)}(1-z^2/n^2) = \lim_{n\to\infty}\frac{n^{-\alpha}(n+1)_\alpha}{\Gamma(1+\alpha)}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\middle|\, z^2/4n^2\right)$$

$$= [\Gamma(1+\alpha)]^{-1}\,{}_0F_1(\ ,1+\alpha,-z^2/4) = (z/2)^{-\alpha}J_\alpha(z) \quad (22)$$

This is a special case of the following theorem

Theorem 2 *Let*

$$\sum_{k=0}^{\infty} b_k z^k < \infty, \qquad |z| < R \tag{23}$$

Then

$$G(z,\nu,\lambda) = \sum_{k=0}^{\infty}\frac{b_k(-\nu)_k(\nu+\lambda)_k}{(k!)^2}\left(-\frac{z}{\nu(\nu+\lambda)}\right)^k, \qquad |z| < |\nu(\nu+\lambda)|R, \tag{24}$$

and $G(z,\nu,\lambda)$ can be rearranged in descending powers of $\nu(\nu+\lambda)$ to yield

$$G(z,\nu,\lambda) = \sum_{j=0}^{\infty} h_j(z,\lambda)[-\nu(\nu+\lambda)]^{-j}, \qquad |z| < |\nu(\nu+\lambda)|R, \tag{25}$$

where the $h_j(z,\lambda)$ are polynomials in λ of order j whose coefficients are entire functions of z Further, for $j \geqslant 1$, $h_j(z,\lambda)$ can be expressed in terms of the derivatives of $h_0(z,\lambda) = h(z)$

The proof is much akin to Theorem 1 and we omit details To construct $h_j(z,\lambda)$, we write

$$h_j(z,\lambda) = \sum_{k=0}^{\infty}\frac{b_k c_{j,k}(\lambda)z^k}{(k!)^2} \tag{26}$$

Now

$$\frac{(-)^k(-\nu)_k(\nu+\lambda)_k}{[\nu(\nu+\lambda)]^k} = (1-k\nu)^{-1}\prod_{j=0}^{k}(1-j\nu)\left(1+\frac{k}{\nu+\lambda}\right)^{-1}\prod_{j=0}^{k}\left(1+\frac{j}{\nu+\lambda}\right)$$

$$= \left(1+\frac{k(k+\lambda)}{(-\nu)(\nu+\lambda)}\right)^{-1}\prod_{j=0}^{k}\left(1+\frac{j(j+\lambda)}{(-\nu)(\nu+\lambda)}\right)$$

$$= \sum_{j=0}^{k} c_{j,k}(\lambda)[(-\nu)(\nu+\lambda)]^{-j} \tag{27}$$

Multiply both sides of (27) by $(\nu - k)(\nu + \lambda + k)[\nu(\nu + \lambda)]^{-1}$, and so obtain the recursion formula

$$c_{j,k+1}(\lambda) - c_{j,k}(\lambda) = k(k + \lambda)c_{j-1,k}(\lambda), \qquad k, j \geqslant 0, \tag{28}$$

and by summation

$$c_{j+1,k}(\lambda) = \sum_{m=0}^{k-1} m(m + \lambda)c_{j,m}(\lambda), \qquad c_{0,m}(\lambda) = 1, \quad m \geqslant 0. \tag{29}$$

Employing the same type of factorization as used to write $g_j(z)$ in terms of the derivatives of $g_0(z)$, we have for the first three $c_{j,k}(\lambda)$,

$$\begin{aligned} c_{0,k}(\lambda) &= 1, \qquad c_{1,k}(\lambda) = \tfrac{1}{6}[k(k-1)][2(k-2)+3] + \tfrac{1}{2}[k(k-1)]\lambda, \\ c_{2,k}(\lambda) &= \frac{1}{360}[k(k-1)(k-2)][20(k-3)(k-4)(k-5) \\ &\quad + 204(k-3)(k-4) + 495(k-3) + 240] \\ &\quad + \frac{1}{6}[k(k-1)(k-2)][(k-3)(k-4) + 6(k-4) + 6]\lambda \\ &\quad + \frac{1}{24}[k(k-1)(k-2)][3(k-3) + 8]\lambda^2. \end{aligned} \tag{30}$$

Thus,

$$\begin{aligned} h_0(z) &= h(z) = \sum_{k=0}^{\infty} b_k z^k/(k!)^2, \qquad h_1(z) = (1 + \lambda)(z^2/2)h^{(2)}(z) + (z^3/3)h^{(3)}(z), \\ h_2(z) &= (2 + 3\lambda + \lambda^2)(z^3/3)h^{(3)}(z) + (11 + 8\lambda + \lambda^2)(z^4/8)h^{(4)}(z) \\ &\quad + (17 + 5\lambda)(z^5/30)h^{(5)}(z) + (z^6/18)h^{(6)}(z). \end{aligned} \tag{31}$$

Note that Theorems 1 and 2 are related by the fact that when $b_k = k!\, a_k$,

$$\lim_{\lambda\to\infty} G(z, \nu, \lambda) = F(z, -\nu). \tag{32}$$

If in Theorem 2,

$$b_k = (\alpha_p)_k(1)_k/(\rho_q)_k, \tag{33}$$

then

$$\begin{aligned} G(z, \nu, \lambda) &= {}_{p+2}F_q\left(\begin{matrix} -\nu, \nu + \lambda, \alpha_p \\ \rho_q \end{matrix}\middle| -\frac{z}{\nu(\nu + \lambda)}\right), \\ h(z) &= {}_pF_q\left(\begin{matrix} \alpha_p \\ \rho_q \end{matrix}\middle| z\right), \qquad h^{(m)}(z) = \frac{(\alpha_p)_m}{(\rho_q)_m}\, {}_pF_q\left(\begin{matrix} \alpha_p + m \\ \rho_q + m \end{matrix}\middle| z\right), \end{aligned} \tag{34}$$

and the expansion (25) is valid when

$$p = q - 1 \quad |z| < \nu(\nu + \lambda), \qquad p < q - 1, \quad |z| < \infty \tag{35}$$

We next turn our attention to three theorems dealing with asymptotic confluent expansions We omit proofs and refer the reader to the work of Fields (1966) The following is a generalization of the statement (8)

Theorem 3 *Let*

$$f(z) = \sum_{k=0}^{\infty} c_k z^k < \infty, \qquad |z| < R \tag{36}$$

Then

$$T(z, \beta) = \sum_{k=0}^{\infty} [c_k/(\beta)_k](\beta z)^k \tag{37}$$

converges for all $z, \beta \neq 0, -1, -2, \ldots$, *and*

$$T(z, \beta) \sim \sum_{j=0}^{\infty} f_j(z)(-\beta)^{-j} \qquad |\beta| \to \infty, \quad |\arg \beta| \leqslant \pi - \delta, \quad 0 < \delta \leqslant \pi/2,$$

$$|z| < R \quad \text{if} \quad |\arg \beta| \leqslant \pi/2,$$

$$|z| < |\sin(\arg \beta)| R \quad \text{if} \quad \pi/2 \leqslant |\arg \beta| \leqslant \pi - \delta, \tag{38}$$

where the $f_j(z)$ *are analytic functions in* $|z| < R$ *and are given by* (41) *below*

For $j \geqslant 1$, $f_j(z)$ can be expressed in terms of the derivatives of $f_0(z) = f(z)$ Let us write

$$\beta^k (\beta)_k = \prod_{j=0}^{k-1} (1 + j\beta)^{-1} = \sum_{j=0}^{\infty} \gamma_{j,k} \beta^{-j} \tag{39}$$

From 2 11(12)

$$\gamma_{j,k} = \frac{(-)^j (k)_j}{j!} B_j^{(1-k)} \qquad j + k > 0,$$

$$= 1, \qquad j = k = 0 \tag{40}$$

Also

$$f_j(z) = \sum_{k=0}^{\infty} c_k (-)^j \gamma_{j,k} z^k = \sum_{k=0}^{\infty} \frac{c_k (k)_j B_j^{(1-k)}}{j!} z^k \tag{41}$$

Now $(k)_j B_j^{(1-k)}$ is a polynomial in k of degree $(2j)$ and vanishes for $k = 0$ and $k = 1$. As in Theorem 1, let Δ denote the forward difference operator with respect to k. If m is a positive integer or zero, the relation

$$\Delta^m(k-1-r)_r = (-r)_m(k-1-r+m)_{r-m} \tag{42}$$

implies that for $j \geqslant 1$,

$$(k)_j B_j^{(1-k)} = \sum_{r=0}^{2j-1} \frac{(k-r-1)_{r+2}}{r!} \left[\Delta^r \left(\frac{(k)_j B_j^{(1-k)}}{k(k-1)}\right)\right]_{k=2}, \qquad j \geqslant 1. \tag{43}$$

Thus,

$$f_j(z) = \sum_{r=0}^{2j-2} \frac{z^{r+2}}{j!\,r!} f_0^{(r+2)}(z) \left[\Delta^r \left(\frac{(k)_j B_j^{(1-k)}}{k(k-1)}\right)\right]_{k=2}, \qquad j \geqslant 1. \tag{44}$$

The first few $f_j(z)$ are as follows:

$$\begin{aligned}
f_0(z) &= f(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad f_1(z) = (z^2/2)f^{(2)}(z),\\
f_2(z) &= (z^2/2)f^{(2)}(z) + (2z^3/3)f^{(3)}(z) + (z^4/8)f^{(4)}(z),\\
f_3(z) &= (z^2/2)f^{(2)}(z) + 2z^3 f^{(3)}(z) + (3z^4/2)f^{(4)}(z)\\
&\quad + (z^5/3)f^{(5)}(z) + (z^6/48)f^{(6)}(z),\\
f_4(z) &= (z^2/2)f^{(2)}(z) + (14z^3/3)f^{(3)}(z) + (61z^4/8)f^{(4)}(z) + (62z^5/15)f^{(5)})(z)\\
&\quad + (131z^6/144)f^{(6)}(z) + (z^7/12)f^{(7)}(z) + (z^8/384)f^{(8)}(z).
\end{aligned} \tag{45}$$

It follows that (8) is the limiting form of the asymptotic expansion,

$$\begin{aligned}
{}_pF_{q+1}\left(\begin{matrix}\alpha_p\\ \rho_q, \beta\end{matrix}\middle|\, \beta z\right) &= {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\, z\right) - \frac{A_2(z)}{2\beta} + \frac{1}{24\beta^2}\{12A_2(z) + 16A_3(z) + 3A_4(z)\}\\
&\quad - \frac{1}{48\beta^3}\{24A_2(z) + 96A_3(z) + 72A_4(z) + 16A_5(z) + A_6(z)\}\\
&\quad + \frac{1}{5760\beta^4}\{2880A_2(z) + 26880A_3(z)\\
&\quad + 43920A_4(z) + 23808A_5(z) + 5240A_6(z)\\
&\quad + 480A_7(z) + 15A_8(z)\} + O(\beta^{-5}),\\
&\quad |\beta| \to \infty, \quad p \leqslant q, \quad |\arg\beta| \leqslant \pi - \delta, \quad 0 < \delta \leqslant \pi/2,\\
&\quad p = q+1, \quad |z| < 1 \qquad \text{if} \quad R(\beta) \geqslant 0
\end{aligned} \tag{46}$$

and

$$|z| < \sin\delta, \quad 0 < \delta \leqslant \pi/2 \qquad \text{if} \quad R(\beta) \leqslant 0,$$

where $A_r(z)$ is defined as in (21)

If n is an integer $\geqslant 0$, the functions $F(z, -n)$ and $G(z, n, \lambda)$ defined in Theorems 1 and 2 are polynomials in z of degree n and so are always well defined Under weaker conditions than stated, Theorems 1 and 2 yield the asymptotic expansion of $F(z, -n)$ and $G(z, n, \lambda)$, respectively, as $n \to \infty$ These results are given by Theorem 4

Theorem 4. *Let*

$$\sum_{k=0}^{\infty} \frac{a_k}{k!} z^k < \infty, \quad |z| < S, \qquad \sum_{k=0}^{\infty} \frac{b_k}{(k!)^2} z^k < \infty, \quad |z| < S \tag{47}$$

Then for n an integer, $n \geqslant 0$,

$$F(z, -n) = \sum_{k=0}^{n} \frac{a_k(-n)_k}{k!} \left(-\frac{z}{n}\right)^k \sim \sum_{j=0}^{\infty} g_j(z)(-n)^{-j},$$
$$n \to \infty, \qquad |z| < S, \tag{48}$$

and

$$G(z, n, \lambda) = \sum_{k=0}^{n} \frac{b_k(-n)_k(n+\lambda)_k}{(k!)^2} \left(-\frac{z}{n(n+\lambda)}\right)^k \sim \sum_{j=0}^{\infty} h_j(z, \lambda)[-n(n+\lambda)]^{-j},$$
$$n \to \infty, \qquad |z| < S, \tag{49}$$

where the $g_j(z)$ are analytic in $|z| < S$ and are defined by (16), *and where the $h_j(z, \lambda)$ are analytic in $|z| < S$ and are defined by* (26)

Another interesting asymptotic confluent expansion is given by Theorem 5

Theorem 5. *Let*

$$v(z) = \sum_{k=0}^{\infty} d_k z^k < \infty, \qquad |z| < R \tag{50}$$

Then

$$S(z, \sigma, a, b) = \sum_{k=0}^{\infty} \frac{d_k(\sigma + a)_k z^k}{(\sigma + b)_k} < \infty,$$
$$|z| < R \qquad \sigma + b \neq 0, -1, -2, \ldots, \tag{51}$$

and can be rearranged in the region $|z| < R/2$ *to give the expansion*

$$S(z, \sigma, a, b) = \sum_{j=0}^{\infty} \frac{(b-a)_j(-z)^j}{(\sigma+b)_j j!} v^{(j)}(z), \qquad |z| < R/2. \tag{52}$$

However, if a and b are bounded, then (52) *is valid asymptotically in the larger region* $|z| < R$. *Thus, if n is an integer* >0,

$$S(z, \sigma, a, b) = \sum_{j=0}^{n-1} \frac{(b-a)_j(-z)^j}{(\sigma+b)_j j!} v^{(j)}(z) + O(\sigma^{-n}),$$

$$|\sigma| \to \infty, \qquad |\arg(\sigma+b)| \leqslant \pi - \delta, \qquad \delta > 0, \qquad |z| < R. \tag{53}$$

If $[(\sigma+b)_j]^{-1}$ *is expanded in reciprocal powers of* σ, *then* (53) *is a confluent expansion in* $1/\sigma$.

Note that if $(b-a) = -(m-1)$, m a positive integer, (52) terminates and is valid for $|z| < R$, a result noticed for hypergeometric series by C. Fox (1927). Theorem 5 includes Kummer's formula 3.8(3). For another special case of Theorem 5, see 9.1(34).

3.6. Integral Representations

We first prove Euler's formula

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\, dt,$$

$$R(c) > R(a) > 0, \qquad |\arg(1-z)| < \pi. \tag{1}$$

The series on the left converges if $|z| < 1$, but the integral on the right is single valued and analytic if $|\arg(1-z)| < \pi$, and so the integral gives the analytic continuation of the ${}_2F_1$. If $|zt| < 1$, the binomial expansion for $(1-zt)^{-b}$ is uniformly convergent and with the aid of 2.6(1), (1) follows by termwise integration. In the integrand of (1) replace t by $t/(1+t)$. Then

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^\infty t^{a-1}(1+t)^{b-c}(1+t-zt)^{-b}\, dt,$$

$$R(c) > R(a) > 0, \qquad |\arg(1-z)| < \pi, \tag{2}$$

or

$${}_2F_1\left(\begin{matrix} a, b \\ a+b+1-c \end{matrix} \middle| 1-z\right)$$

$$= \frac{\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(b+1-c)} \int_0^\infty t^{a-1}(1+t)^{c-a-1}(1+zt)^{-b}\, dt,$$

$$R(a) > 0, \qquad R(b+1-c) > 0, \qquad |\arg z| < \pi. \tag{3}$$

From the differential equation satisfied by the ${}_2F_1$ [see 3 7(1)] the formula

$$(1-z)[\delta(\delta+c-1)-z(\delta+a)(\delta+b)][t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}]$$
$$= -b(\partial/\partial t)[t^a(1-t)^{c-a}(1-zt)^{-b-1}] \tag{4}$$

furnishes an alternative proof of (1) and (3) More generally, 3 7(1) is satisfied by

$$\int_C f(t)\,dt, \qquad f(t) = t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} \tag{5}$$

if C is either closed on the Riemann surface of the integrand or terminates at the zeros of $t^a(1-t)^{c-a}(1-zt)^{-b-1}$ Expand $(1-zt)^{-b}$ by the binomial series and use the contour integrals for the beta function (see 2 7) to get

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| z\right) = \frac{i\Gamma(c)e^{i\pi(a-c)}}{2\Gamma(a)\Gamma(c-a)\sin\pi(c-a)}\int_0^{(1+)} f(t)\,dt, \tag{6}$$

if $R(b) > 0$, $(c-a)$ is not a positive integer, $|\arg(1-z)| < \pi$,

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| z\right) = -\frac{i\Gamma(c)e^{-i\pi a}}{2\Gamma(a)\Gamma(c-a)\sin\pi a}\int_1^{(0+)} f(t)\,dt, \tag{7}$$

if $R(c) > R(a)$, b is not a positive integer, $|\arg(-z)| < \pi$,

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| z\right) = -\frac{\Gamma(c)e^{-i\pi c}}{4\Gamma(a)\Gamma(c-a)\sin\pi a\sin\pi(c-a)}\int^{(1+,0+,1-,0-)} f(t)\,dt, \tag{8}$$

if none of the numbers a, $1-c$, $c-a$ is a positive integer, $|\arg(-z)| < \pi$ In (6)–(8), the path of integration begins at a point of the Riemann surface of $f(t)$ [see (5)], t real, $0 \leqslant t \leqslant 1$, and t^a, $(1-t)^{c-a}$ denote the principal values of these functions Also $(1-zt)^{-b}$ is such that it approaches 1 if z approaches 0

The following are generalizations of (1)

$${}_{p+1}F_{q+1}\left(\begin{matrix} \beta, \alpha_p \\ \beta+\sigma, \rho_q \end{matrix}\middle| z\right)$$
$$= \frac{\Gamma(\beta+\sigma)z^{1-\beta-\sigma}}{\Gamma(\beta)\Gamma(\sigma)}\int_0^z t^{\beta-1}(z-t)^{\sigma-1}\,{}_pF_q\left(\begin{matrix} \alpha_p \\ \rho_q \end{matrix}\middle| t\right)dt \tag{9}$$

$$= \frac{\Gamma(\beta+\sigma)}{\Gamma(\beta)\Gamma(\sigma)}\int_0^1 t^{\beta-1}(1-t)^{\sigma-1}\,{}_pF_q\left(\begin{matrix} \alpha_p \\ \rho_q \end{matrix}\middle| zt\right)dt \tag{10}$$

$$= \frac{2\Gamma(\beta+\sigma)}{\Gamma(\beta)\Gamma(\sigma)}\int_0^{\pi/2}(\sin\theta)^{2\beta-1}(\cos\theta)^{2\sigma-1}\,{}_pF_q\left(\begin{matrix} \alpha_p \\ \rho_q \end{matrix}\middle| z\sin^2\theta\right)d\theta \tag{11}$$

$$= \frac{2\Gamma(\beta+\sigma)}{\Gamma(\beta)\Gamma(\sigma)}\int_0^{\pi/2}(\cos\theta)^{2\beta-1}(\sin\theta)^{2\sigma-1}\,{}_pF_q\left(\begin{matrix} \alpha_p \\ \rho_q \end{matrix}\middle| z\cos^2\theta\right)d\theta. \tag{12}$$

under the conditions

$$p \leqslant q+1, \quad R(\beta) > 0, \quad R(\sigma) > 0; \qquad |z| < 1 \quad \text{if} \quad p = q+1.$$

Integrals like the above with ${}_pF_q$ replaced by an arbitrary function are sometimes called beta transforms. Representations for the ${}_{p+1}F_{q+1}$ above may be written as loop integrals [see (20), (21)].

The Laplace transform and the inverse Laplace transform of a ${}_pF_q$ are also members of the hypergeometric family. Thus,

$${}_{p+1}F_q\left(\begin{matrix}\sigma, \alpha_p \\ \rho_q\end{matrix}\middle| \omega/z\right) = \frac{z^\sigma}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1}\, {}_pF_q\left(\begin{matrix}\alpha_p \\ \rho_q\end{matrix}\middle| \omega t\right) dt \tag{13}$$

is valid under the five cases listed below. In each case $R(\sigma) > 0$ and $z \neq 0$. It is convenient to set down the following conditions:

$$R(\sigma - \alpha_j) < 1, \qquad j = 1, 2, \ldots, p, \tag{14}$$

$$\nu = \sum_{h=1}^{p} \alpha_h - \sum_{h=1}^{q} \rho_h . \tag{15}$$

CASE 1. $p < q$, $|\arg z| < \pi/2$.

CASE 2. $p = q-1$, $|\arg z| = \pi/2$, $\arg \omega = \pi$, (14), $R(4\sigma + 2\nu) < 3$.

CASE 3. $p = q$, $|\arg z| < \pi/2$, $R(z) > R(\omega)$ or $R(z) = R(\omega)$, $z \neq \omega$ and $R(\sigma + \nu) < 1$.

CASE 4. $p = q$, $|\arg z| = \pi/2$, $\pi/2 < |\arg \omega| < 3\pi/2$, (14).

CASE 5. $p = q$, $|\arg z| = |\arg \omega| = \pi/2$, $z \neq \omega$, (14), $R(\sigma + \nu) < 1$.

In Cases 3 and 5, we can have $z = \omega$ if $R(\sigma + \nu) < 0$.

The above conditions arise to insure that the integral is convergent and, aside from the condition $R(\sigma) > 0$, are derived using the asymptotic properties of the ${}_pF_q$ for large argument [see 5.11 and the statement 5.3(3)]. A generalization of (13) is given by 5.6.3(1). Using the latter, we find that for $zy \neq 0$,

$$\int_0^\infty e^{-zt} t^{\sigma-1}\, {}_pF_{p-1}\left(\begin{matrix}\alpha_p \\ \rho_{p-1}\end{matrix}\middle| -yt\right) dt = \frac{\Gamma(\rho_{p-1}) z^{-\sigma}}{\Gamma(\alpha_p)} G^{p+1,1}_{r,p+1}\left(\frac{z}{y}\middle| \begin{matrix}1, \rho_{p-1} \\ \sigma, \alpha_p\end{matrix}\right),$$

$$R(\sigma) > 0, \quad |\arg y| < \pi, \quad |\arg z| < \pi/2 \qquad \text{or}$$

$$|\arg z| = \pi/2 \quad \text{and} \quad R(\sigma - \alpha_j) < 1, \quad j = 1, 2, \ldots, p. \tag{16}$$

In (13) and (16) we supposed that $z \neq 0$ By appropriate use of limiting forms we can let $z \to 0$ Thus

$$\int_0^\infty t^{\sigma-1}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\,-\eta t\right)dt = \frac{\eta^{-\sigma}\Gamma(\sigma)\Gamma(\alpha_p-\sigma)\Gamma(\rho_q)}{\Gamma(\alpha_p)\Gamma(\rho_q-\sigma)},$$

$$0 < R(\sigma) < R(\alpha_j) \qquad j = 1\ 2 \quad p \qquad \eta \neq 0$$

$$q = p + 1 \qquad R(\nu + 2\sigma) < \tfrac{1}{2} \qquad \eta > 0$$

$$q \quad p \qquad |\arg\eta| < \pi/2 \quad \text{or} \quad |\arg\eta| = \pi/2 \quad \text{if} \quad R(\nu+\sigma) < 1 \tag{17}$$

$$q = p \quad 1 \qquad |\arg\eta| < \pi$$

where ν is given by (15)

We also have for $R(\sigma) > 0$ and $z \neq 0$,

$${}_p\,{}_2F_q\left(\begin{matrix}\sigma\ \ \sigma+\frac{1}{2}\ \ \alpha_p\\ \rho_q\end{matrix}\middle|\,4\omega^2/z^2\right) = \frac{z^\sigma}{\Gamma(\sigma)}\int_0^\infty e^{-zt}t^{\sigma-1}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\,\omega^2t^2\right)dt$$

$$p \leqslant q \quad 2 \qquad |\arg z| < \pi/2$$

$$p = q - 1 \qquad \text{(a)} \qquad R(z) > 2|R(\omega)| \geqslant 0 \qquad \text{or}$$

$$\text{(b)} \qquad R(z) = 2|R(\omega)| > 0 \tag{18}$$

$$R(\sigma+\nu) < -\tfrac{1}{2} \qquad \text{or}$$

$$\text{(c)} \qquad R(z) = R(\omega) = 0$$

$$R(\sigma+\nu) < -\tfrac{1}{2}$$

$$R(\sigma \quad 2\alpha_h) < 1 \qquad h \quad 1\ 2 \quad p$$

where ν is as in (15)

The inverse Laplace transform is given by

$$\omega^\beta\,{}_1F_q\left(\begin{matrix}\alpha_p\\ \beta\ \rho_q\end{matrix}\middle|\,\omega z\right) = \frac{\Gamma(\beta)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\omega t}t^{-\beta}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\,z/t\right)dt$$

$$\omega \text{ real} \qquad \omega \neq 0 \qquad R(\beta) > 0 \qquad c > 0 \qquad p \leqslant q \tag{19}$$

Also $|\arg(1 - z/c)| < \pi$ if $p = q + 1$

Next we turn to some loop and other integrals involving the ${}_pF_q$ From 2 7(9)

$${}_{p+1}F_{q+1}\left(\begin{matrix}\alpha_p\ \ \beta\\ \rho_q\ \ \sigma+\beta\end{matrix}\middle|\,z\right) = \frac{\Gamma(\sigma+\beta)\Gamma(1-\sigma)}{2\pi i\Gamma(\beta)}\int_0^{(1+)} t^{\beta-1}(t \quad 1)^{\sigma-1}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\,zt\right)dt$$

$$R(\beta) > 0 \quad p \leqslant q \qquad p \quad q+1 \quad \text{if} \quad |z| < 1 \tag{20}$$

and so also

$$
{}_{p+1}F_{q+1}\left(\begin{matrix}\alpha_p, \beta\\ \rho_q, \sigma+\beta\end{matrix}\middle| z^{-1}\right) = -\frac{\Gamma(\sigma+\beta)\Gamma(1-\sigma)}{2\pi i\Gamma(\beta)}
$$

$$
\times \int_{\infty}^{(1+)} t^{-\sigma-\beta}(1-t)^{\sigma-1}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| (zt)^{-1}\right) dt,
$$

$$
R(\beta) > 0, \quad p \leqslant q; \qquad p = q+1 \quad \text{if} \quad |z| < 1. \tag{21}
$$

Further loop integrals may be derived using 2.7(8, 10). The following loop integral is easily proved from 2.7(3). Thus under the conditions given for (13) but without $R(\sigma) > 0$,

$$
{}_{p+1}F_q\left(\begin{matrix}\sigma, \alpha_p\\ \rho_q\end{matrix}\middle| \omega/z\right) = -\frac{\Gamma(1-\sigma)z^{\sigma}}{2\pi i}\int_{\infty}^{(0+)} e^{-zt}(-t)^{\sigma-1}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| \omega t\right) dt. \tag{22}
$$

Similarly, from 2.7(1),

$$
\omega^{\beta-1}\,{}_pF_{q+1}\left(\begin{matrix}\alpha_p\\ \rho_q, \beta\end{matrix}\middle| \omega z\right) = \frac{\Gamma(\beta)}{2\pi i}\int_{-\infty}^{(0+)} e^{\omega t}t^{-\beta}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z/t\right) dt, \tag{23}
$$

where $\omega > 0$ and $p \leqslant q$. Equation (23) is also valid if $p = q+1$ provided that the radius of the circular part of the contour $(-\infty, 0+)$ in the t-plane is greater than $|z|$. This radius is arbitrary if $p \leqslant q$. This radius is arbitrary if $p \leqslant q$.

The integral

$$
\int_{\infty}^{(0+)} (-t)^{\sigma-1}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| -\eta t\right) dt = -\frac{2\pi i\eta^{-\sigma}\Gamma(\alpha_p-\sigma)\Gamma(\rho_q)}{\Gamma(1-\sigma)\Gamma(\alpha_p)\Gamma(\rho_q-\sigma)}, \tag{24}
$$

is valid under the same conditions as for (17), but without the restriction $R(\sigma) > 0$. Also, for $\eta \neq 0$,

$$
\int_{-c-i\infty}^{-c+i\infty} (-t)^{\sigma-1}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| -\eta t\right) dt = \frac{2\pi i\eta^{-\sigma}\Gamma(\alpha_p-\sigma)\Gamma(\rho_q)}{\Gamma(1-\sigma)\Gamma(\alpha_p)\Gamma(\rho_q-\sigma)},
$$

$$
\begin{aligned}
&c > 0, && R(\sigma-\alpha_h) < 0, && h = 1, 2, \ldots, p,\\
&q = p, && \eta > 0, && R(\sigma+\nu) < 1,\\
&q = p-1, && |\arg\eta| < \pi/2, && c < R(1/\eta),
\end{aligned} \tag{25}
$$

where ν is defined by (15), and under these same conditions

$$
\int_{-c-i\infty}^{-c+i\infty} (-t)^{\sigma-1}\,{}_{f+2}F_{g+1}\left(\begin{matrix}-n, n+\lambda, \gamma_f\\ \beta+1, \delta_g\end{matrix}\middle| -\frac{\omega}{t}\right){}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| -\eta t\right) dt
$$

$$
= \frac{2\pi i\eta^{-\sigma}\Gamma(\alpha_p-\sigma)\Gamma(\rho_q)}{\Gamma(1-\sigma)\Gamma(\alpha_p)\Gamma(\rho_q-\sigma)}\,{}_{p+f+2}F_{q+g+2}\left(\begin{matrix}-n, n+\lambda, \gamma_f, \alpha_p-\sigma\\ \beta+1, \delta_g, 1-\sigma, \rho_q-\sigma\end{matrix}\middle| \omega\eta\right), \tag{26}
$$

$$\int_{-c-i\infty}^{c+i\infty}(-t)^{\sigma-1}\,{}_{f+1}F_{g+1}\left(\begin{matrix}-n,\gamma_f\\ \beta+1\ \delta_g\end{matrix}\middle|-\frac{\omega}{t}\right){}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|-\eta t\right)dt$$
$$=\frac{2\pi i\eta^{-\sigma}\Gamma(\alpha_p-\sigma)\Gamma(\rho_q)}{\Gamma(1-\sigma)\Gamma(\alpha_p)\Gamma(\rho_q-\sigma)}\,{}_{p+f+1}F_{q+g+2}\left(\begin{matrix}-n,\gamma_f\ \alpha_p-\sigma\\ \beta+1,\delta_g,1-\sigma,\rho_q-\sigma\end{matrix}\middle|\omega\eta\right) \tag{27}$$

For generalizations of Eqs (9)–(27) see 5 6

One of the most useful representations for the ${}_2F_1$, and indeed for the ${}_pF_q$ as well as a generalization of the latter [see 5 2(1)], is an integral of the Mellin–Barnes type We consider

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\middle|z\right)=(2\pi i)^{-1}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)(-z)^t}{\Gamma(c+t)}\,dt,$$
$$|\arg(-z)|<\pi, \tag{28}$$

where the path of integration is indented so that the poles due to $\Gamma(-t)$ lie to the right of the path while those due to $\Gamma(a+t)$ and $\Gamma(b+t)$ lie to the left of the path Such a contour is always possible provided that neither a nor b is a negative integer or zero We further suppose that $(b-a)$ is not an integer or zero so that the poles are simple For the case when this does not hold, see the discussion surrounding 3 10(14, 15)

If U is the integrand of (28), we can write, in virtue of 2 2(2),

$$U=-\frac{\pi\Gamma(a+t)\Gamma(b+t)(-z)^t}{\Gamma(c+t)\Gamma(1+t)\sin\pi t}$$

Let $t=u+iv$ on the path of integration Then

$$\left|\frac{(-z)^t}{\sin\pi t}\right|=\left|\frac{|z|^t\exp[it\arg(-z)]}{\sin\pi u\cosh\pi v+i\cos\pi u\sinh\pi v}\right|$$
$$=\frac{\exp[u(\ln|z|)-v\arg(-z)]}{\{\sin^2\pi u\cosh^2\pi v+\cos^2\pi u\sinh^2\pi v\}^{1/2}}$$

Using 2 11(11), we see that

$$|U|=O(|t^{a+b-c-1}|\exp[u(\ln|z|)]\exp\{-|v|[\pi\pm\arg(-z)]\}) \tag{29}$$

as $t\to\pm\infty$ on the contour Thus U is an analytic function of z provided that $|\arg(-z)|\leqslant\pi-\epsilon$ where ϵ is arbitrarily small but positive

Now let

$$I=(2\pi i)^{-1}\int_C U\,dt$$

where C is a semicircle to the right of the imaginary axis with center at the origin and radius $N+\frac{1}{2}$, N a large positive integer Let

$t = (N + \frac{1}{2})e^{i\theta}$ on C. Then by an analysis similar to that which led to (29), we have

$$| U | = O((N + \tfrac{1}{2})^{R(a+b-c-1)} \exp\{(N + \tfrac{1}{2})[\cos\theta \ln| z | - \epsilon| \sin\theta |]\})$$

as $N \to \infty$. If $| z | < 1$ so that $\ln | z |$ is negative, since $-\pi/2 \leqslant \theta \leqslant \pi/2$ and $\epsilon > 0$, it follows that U is an exponential decay whence

$$\lim_{N\to\infty} | V | = 0.$$

By Cauchy's theorem

$$\int_{-i\infty}^{i\infty} U\,dt - \left\{\int_{-i\infty}^{-(N+\frac{1}{2})i} U\,dt + \int_C U\,dt + \int_{(N+\frac{1}{2})i}^{i\infty} U\,dt\right\}$$

equals the negative of $(2\pi i)$ times the residues of U at the points $t = 0, 1,\ldots, N$. As $N \to \infty$ the last three integrals tend to zero. Since

$$\lim_{t\to N} (t - N)U = -\frac{\pi\Gamma(a + N)\Gamma(b + N)(-z)^N}{\Gamma(c + N)N!\,(d/dt)(\sin \pi t)|_{t=N}} = \frac{\Gamma(a + N)\Gamma(b + N)z^N}{\Gamma(c + N)N!},$$

we get 3.1(13) for $| z | < 1$. This last restriction may now be removed by the principle of analytic continuation.

Next suppose that

$$W = (2\pi i)^{-1} \int_D U\,dt$$

where D is a semicircle to the left of the imaginary axis with center at the origin and radius $N + \delta$, N a large positive integer, and δ is such that none of the poles of $\Gamma(a + t)$ and $\Gamma(b + t)$ lie on D. Then by an analysis used to prove (28), we get

$$\frac{\Gamma(a)\Gamma(b - a)}{\Gamma(c - a)}(-z)^{-a}\,{}_2F_1\left(\begin{matrix} a, 1 + a - c \\ 1 + a - b \end{matrix}\middle| z^{-1}\right)$$
$$+ \text{ a like expression with } a \text{ and } b \text{ interchanged}$$
$$= (2\pi i)^{-1} \int_{-i\infty}^{i\infty} \frac{\Gamma(a + t)\Gamma(b + t)\Gamma(-t)(-z)^t}{\Gamma(c + t)}\,dt, \qquad | \arg(-z)| < \pi, \quad (30)$$

where the path of integration is as in (28). Coupling (28) and (30), we get 3.9(1), a formula connecting three ${}_2F_1$'s. This is an important relation for it serves as the analytic continuation of ${}_2F_1(a, b; c; z)$ into the domain where $| z | > 1$. A rather complete discussion of analytic continuation is given in 3.9.

3 7 Differential Equations for the ${}_2F_1$

The unique second order linear differential equation with three isolated regular singularities at $z = 0$ 1 and ∞ can be written in the form

$$[z(1-z)D^2 + \{c - (a+b+1)z\}D - ab]\,w = 0 \qquad D = d/dz \tag{1}$$

or by virtue of 2 9(1)

$$\{\delta(\delta + c - 1) \quad z(\delta + a)(\delta + b)\}\,w - 0 \qquad \delta = zD \tag{2}$$

We call (1) or (2) the Gaussian hypergeometric equation For a discussion of the second order linear differential equation with three isolated regular singularities located at arbitrary points and related data see Erdelyi *et al* (1953) Norlund (1963) and Poole (1960)

Integration of (2) leads to

$$[(\delta + c - 1) - z(\delta + b) - z(a - 1)]\,w - (a-1)(b-1)\int_0^z w(t)\,dt = c - 1 \tag{3}$$

so that if $b = 1$, w satisfies the first order nonhomogeneous differential equation

$$[(\delta + c - 1) - z(\delta + 1) - z(a - 1)]\,w = c - 1 \tag{4}$$

We shall presently show that ${}_2F_1(a\ b\ c\ z)$ is a solution of (1) More generally the differential equation

$$\begin{aligned}\frac{h(h-1)}{(h\ )^2}y'' &+ \left\{\frac{h(h-1)}{(h\ )^3}\left(\frac{2\alpha h}{z} + 2f\,h\ - h''\right) + \frac{(a+b+1)\,h - c}{h}\right\}y \\ &+ \left\{\left(\frac{\alpha}{z} + f\ \right)\left(\frac{(a+b+1)\,h - c}{h}\right) + \frac{h(h-1)}{(h\ )^3}\left[\frac{\alpha(\alpha-1)\,h}{z^2}\right.\right. \\ &\left.\left. + \frac{2\alpha f\ h}{z} + f'h\ + (f\ )^2 h \qquad \frac{\alpha h''}{z} \quad f\,h''\right] + ab\right\}y = 0\end{aligned} \tag{5}$$

with $h = h(z)$ $f = f(z)$ and $y \equiv y(z)$ is satisfied by

$$y(z) = z^{-\alpha z\ f(z)}\ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle|\, h(z)\right) \tag{6}$$

By the method of Frobenius assume the solution of (1) in the form

$$w = \sum_{k=0}^{\infty} u_k z^{k\ m} \tag{7}$$

Combine (2) with (7) and employ 2.9(3). Then

$$\sum_{k=0}^{\infty} u_k\{(m+k)(m+k+c-1)\,z^k - (m+k+a)(m+k+b)\,z^{k+1}\} = 0. \quad (8)$$

The coefficient of $u_0 z^0$ set to zero is the indicial equation which has two zeros, which we suppose are distinct. Denote these by m_h, $h = 0, 1$, so that $m_0 = 0$ and $m_1 = 1 - c$. Equating like powers of z in (8), we get the recursion relation

$$u_{k+1} = \frac{(k+m_h+a)(k+m_h+b)}{(k+m_h+c)(k+m_h+1)}\,u_k, \quad (9)$$

and so

$$u_k = \frac{(m_h+a)_k\,(m_h+b)_k}{(m_h+c)_k\,(m_h+1)_k}\,u_0. \quad (10)$$

For each m_h, we can take $u_0 = 1$ and so the linearly independent solutions of (1) in the vicinity of the origin are proportional to

$$w_1 = {}_2F_1(a, b; c; z), \quad (11)$$

and

$$w_2 = z^{1-c}\,{}_2F_1(1+a-c, 1+b-c; 2-c; z), \quad (12)$$

provided that c is not an integer or zero. Each series converges for $|z| < 1$, and appeal to 3.3(3) shows that w_1 and w_2 are

$$\begin{aligned} &\text{absolutely convergent for } |z| = 1 \text{ if } R(a+b-c) < 0,\\ &\text{conditionally convergent for } |z| = 1,\ z \neq 1 \text{ if } 0 \leqslant R(a+b-c) < 1, \quad (13)\\ &\text{divergent if } |z| = 1 \text{ and } 1 \leqslant R(a+b-c). \end{aligned}$$

It is clear that we must enlarge the definition of the solutions of (1) when c is an integer or zero. First we note that w_1 is a polynomial of degree m, a positive integer or zero if either $a = -m$ or $b = -m$. Throughout our discussion, since w_1 is symmetric in a and b, if one of these is specialized, we will let it be a. If $a = -m$ and c is a negative integer, $m + c < 1$, then a well-defined solution of (1) is

$$\begin{aligned} (a, b; c; z) &= \sum_{k=0}^{-a} \frac{(a)_k (b)_k z^k}{(c)_k k!} = \sum_{k=0}^{m} \frac{(-m)_k (b)_k z^k}{(1-s)_k k!}\\ &= \frac{(s-1-m)!\,(b)_m z^m}{(s-1)!}\,{}_2F_1\left(\begin{matrix}-m, s-m\\ 1-m-b\end{matrix}\,\middle|\, z^{-1}\right),\\ c &= 1-s, \qquad s \text{ a positive integer}, \qquad m < s, \end{aligned} \quad (14)$$

and this is independent of w_2 Likewise, if a and c are positive integers, $a < c$, then

$$z^{1-c} f(1 + a - c, 1 + b - c, 2 - c, z)$$

$$= z^{-s} \sum_{k=0}^{s-m} \frac{(m - s)_k (b - s)_k z^k}{(1 - s)_k k!}$$

$$= \frac{(m - 1)! \, \Gamma(b - m) \, z^{-m}}{(s - 1)! \, \Gamma(b - s)} \, {}_2F_1 \left(\begin{matrix} m - s, m \\ 1 + m - b \end{matrix} \middle| z^{-1} \right), \qquad (15)$$

$a = m$ a positive integer, $c = 1 + s$, s a positive integer or zero, $m < s + 1$, is a well-defined solution of (1) and is independent of w_1 In (14) and (15), the ${}_2F_1$ notation applies only when $(1 - m - b)$ and $(1 + m - b)$, respectively, are not negative integers

If $a = c$, $w_1(z) = (1 - z)^{-b}$ and a second solution is given by (12) provided that a is not a positive integer When $a = c$, it can be shown that a solution of (1) is

$$u = (1 - a)(1 - z)^{-b} \int^z t^{-a}(1 - t)^{b-1} \, dt \qquad (16)$$

In particular, if m and n are positive integers or zero, and the constant of integration in (16) is ignored,

$$u = z^{-n} {}_2(1 - z)^{-1} \sum_{k=0}^{n-m} \frac{(m - n)_k}{(-n)_k} \left(\frac{z}{z - 1} \right)^k,$$

$$c = a = n + 2, \qquad b = n - m + 1, \qquad n \geqslant m, \qquad (17)$$

$$u = (1 - m)(1 - z)^{-m-n} {}_2 \left\{ \sum_{\substack{k=0 \\ k \neq m-1}}^{m+n} \frac{(-)^k \binom{m+n}{k} z^{k+1-m}}{k + 1 - m} + (-)^{m-1} \binom{m+n}{m-1} \ln z \right\},$$

$$c = a = m, \qquad b = m + n + 1 \qquad (18)$$

If c is a negative integer or zero, say $c = -m$, w_1 is undefined In this event, a solution can be taken proportional to

$$\lim_{c \to -m} \frac{w_1}{\Gamma(c)} = \frac{(a)_{m+1}(b)_{m+1} z^{m+1}}{(m + 1)!} \, {}_2F_1(m + a + 1, m + b + 1, m + 2, z) \qquad (19)$$

However, this is proportional to w_2 with $c = -m$ It is convenient to *defer a full investigation of the complete solution of (1) to 3 10*

3.8. Kummer's Solutions

Kummer's 24 solutions of 3.7(1) are as follows:

$$w_1 = {}_2F_1(a, b; c; z) \tag{1}$$
$$= (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b; c; z) \tag{2}$$
$$= (1-z)^{-a}\, {}_2F_1(a, c-b; c; z/(z-1)) \tag{3}$$
$$= (1-z)^{-b}\, {}_2F_1(c-a, b; c; z/(z-1)). \tag{4}$$

$$w_2 = z^{1-c}\, {}_2F_1(1+a-c, 1+b-c; 2-c; z) \tag{5}$$
$$= z^{1-c}(1-z)^{c-a-b}\, {}_2F_1(1-a, 1-b; 2-c; z) \tag{6}$$
$$= z^{1-c}(1-z)^{c-a-1}\, {}_2F_1(1+a-c, 1-b; 2-c; z/(z-1)) \tag{7}$$
$$= z^{1-c}(1-z)^{c-b-1}\, {}_2F_1(1+b-c, 1-a; 2-c; z/(z-1)). \tag{8}$$

$$w_3 = {}_2F_1(a, b; a+b+1-c; 1-z) \tag{9}$$
$$= z^{1-c}\, {}_2F_1(a+1-c, b+1-c; a+b+1-c; 1-z) \tag{10}$$
$$= z^{-a}\, {}_2F_1(a, a+1-c; a+b+1-c; 1-z^{-1}) \tag{11}$$
$$= z^{-b}\, {}_2F_1(b, b+1-c; a+b+1-c; 1-z^{-1}). \tag{12}$$

$$w_4 = (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b, c+1-a-b; 1-z) \tag{13}$$
$$= z^{1-c}(1-z)^{c-a-b}\, {}_2F_1(1-a, 1-b; c+1-a-b; 1-z) \tag{14}$$
$$= z^{a-c}(1-z)^{c-a-b}\, {}_2F_1(c-a, 1-a; c+1-a-b; 1-z^{-1}) \tag{15}$$
$$= z^{b-c}(1-z)^{c-a-b}\, {}_2F_1(c-b, 1-b; c+1-a-b; 1-z^{-1}). \tag{16}$$

$$w_5 = (z^{-1}e^{i\pi})^a\, {}_2F_1(a, a+1-c; a+1-b; z^{-1}) \tag{17}$$
$$= (z^{-1}e^{i\pi})^{c-b}(1-z)^{c-a-b}\, {}_2F_1(1-b, c-b; a+1-b; z^{-1}) \tag{18}$$
$$= (1-z)^{-a}\, {}_2F_1(a, c-b; a+1-b; (1-z)^{-1}) \tag{19}$$
$$= (z^{-1}e^{i\pi})^{c-1}(1-z)^{c-1-a}\, {}_2F_1(1-b, a+1-c; a+1-b; (1-z)^{-1}). \tag{20}$$

$$w_6 = (z^{-1}e^{i\pi})^b\, {}_2F_1(b, b+1-c; b+1-a; z^{-1}) \tag{21}$$
$$= (z^{-1}e^{i\pi})^{c-a}(1-z)^{c-a-b}\, {}_2F_1(1-a, c-a; b+1-a; z^{-1}) \tag{22}$$
$$= (1-z)^{-b}\, {}_2F_1(b, c-a; b+1-a; (1-z)^{-1}) \tag{23}$$
$$= (z^{-1}e^{i\pi})^{c-1}(1-z)^{c-1-b}\, {}_2F_1(1-a, b+1-c; b+1-a; (1-z)^{-1}). \tag{24}$$

The formulas (1)–(4) follow by transformation of the variable in 3.7(1). They may also be proved from the integral representation 3.6(1). There replace t by $(1-u)/(1-uz)$, $1-u$, and $u/(1-z+uz)$ to get (2), (3), and (4), respectively. That (3) and (4) are identical follows since w_1 is symmetric in the parameters a and b.

In 3.7(1) replace z by $(1-z)$ whence we get (1) with c replaced by $(a+b+1-c)$. Thus in the neighborhood of $z=1$, linear independent solutions of 3.7(1) are proportional to (9) and (13) provided $(a+b+1-c)$ is not an integer or zero. As in the case for w_1 and w_2, we get the formulas (9)–(16).

In 3.7(1), replace z by $1/x$ and w by $x^a w$ whence δ must be replaced by $(-\delta+a)$ [see 2.9(2, 4)]. We then get an equation of the type 3.7(1), and in the neighborhood of infinity, linear independent solutions of 3.7(1) are proportional to w_5 and w_6 provided that $(a+1-b)$ is not an integer or zero. We therefore get the eight formulas (17)–(24).

The above 24 solutions of 3.7(1) are known as Kummer's solutions. The conditions for convergence may be inferred from the discussion about 3.7(13) and are summarized in Table 3.1.

TABLE 3.1

Formula	Absolute convergence	Absolute convergence on the boundary*	Conditional convergence on the boundary	Except at
(1), (5)	$\lvert z\rvert < 1$	$R(\alpha) < 0$	$0 \leq R(\alpha) < 1$	$z = 1$
(2), (6)	$\lvert z\rvert < 1$	$R(\alpha) > 0$	$-1 < R(\alpha) \leq 0$	$z = 1$
(3), (7)	$R(z) < \frac{1}{2}$	$R(\beta) < 0$	$0 \leq R(\beta) < 1$	$z = \frac{1}{2}$
(4), (8)	$R(z) < \frac{1}{2}$	$R(\beta) > 0$	$-1 < R(\beta) \leq 0$	$z = \frac{1}{2}$
(9), (13)	$\lvert 1-z\rvert < 1$	$R(\gamma) < 0$	$0 \leq R(\gamma) < 1$	$z = 0$
(10), (14)	$\lvert 1-z\rvert < 1$	$R(\gamma) > 0$	$-1 < R(\gamma) \leq 0$	$z = 0$
(11), (15)	$R(z) > \frac{1}{2}$	$R(\beta) < 0$	$0 \leq R(\beta) < 1$	$z = \infty$
(12), (16)	$R(z) > \frac{1}{2}$	$R(\beta) > 0$	$-1 < R(\beta) \leq 0$	$z = \infty$
(17), (21)	$\lvert z\rvert > 1$	$R(\alpha) < 0$	$0 \leq R(\alpha) < 1$	$z = 1$
(18), (22)	$\lvert z\rvert > 1$	$R(\alpha) > 0$	$-1 < R(\alpha) \leq 0$	$z = 1$
(19), (23)	$\lvert 1-z\rvert > 1$	$R(\gamma) < 0$	$0 \leq R(\gamma) < 1$	$z = 0$
(20), (24)	$\lvert 1-z\rvert > 1$	$R(\gamma) > 0$	$-1 < R(\gamma) \leq 0$	$z = 0$

* $\alpha = a+b-c$, $\beta = a-b$, $\gamma = c-1$.

From Table 3.1 we see that the pairs (3), (4) and (7), (8) furnish the analytic continuation of w_1 and w_2, respectively, from the interior of the unit circle with center at the origin to the half-plane $R(z) < \frac{1}{2}$. Similarly, the pairs (11), (12) and (15), (16) give the analytic continuation

of w_3 and w_4, respectively, from the interior of the unit circle with center at $z = 1$ to the half-plane $R(z) > \frac{1}{2}$, and the pairs (19), (20) and (23), (24) provide the analytic continuation of w_5 and w_6, respectively, from the exterior of the unit circle with center at the origin to the interior of this circle which lies in the left half-plane.

It follows from Kummer's solutions that if c is not an integer or zero, one of the fundamental solutions of 3.7(1) is composed of only a finite number of terms whenever at least one of the numbers a, b, $c - a$, $c - b$ is an integer or zero. This is the same as saying that at least one of the eight numbers $\pm(c - 1) \pm (a - b) \pm (a + b - c)$ is an odd integer. Such solutions are called degenerate.

3.9. Analytic Continuation

The integrals 3.6(1, 6) define single valued analytic functions of z in the domain $|\arg(1 - z)| < \pi$, and so serve for the analytic continuation of the ${}_2F_1$ hypergeometric series into this domain. It is convenient to denote the analytic continuation of the ${}_2F_1$ series by ${}_2F_1$, and this means the principal branch of the analytic function generated by the hypergeometric series. Similarly, the integrals 3.6(7, 8, 28) serve for the analytic continuation of the ${}_2F_1$ series into the domain $|\arg(-z)| < \pi$.

We suppose for the moment that w_1 and w_2, w_3 and w_4, and w_5 and w_6 (see 3.8) are the fundamental solutions of 3.7(1) about $z = 0$, 1, and ∞, respectively. Clearly all six of these quantities cannot be independent, and any three must be linearly related. This gives rise to twenty relations. Let the triplet of numbers (p, q, r) stand for the relation between w_p, w_q, and w_r. Thus (1, 5, 6) signifies the equation connecting w_1, w_5, and w_6. This expression, given by (1) below, has already been proved [see 3.6(28–30)]. In 3.6(30), we have replaced $(-z)^{-a}$ and $(-z)^{-b}$ by $(z^{-1}e^{i\pi})^a$ and $(z^{-1}e^{i\pi})^b$, respectively, so that z is now restricted by $0 < \arg z < 2\pi$. In (1, 5, 6), replace a, b, and c by $1 + a - c$, $1 + b - c$, and $2 - c$, respectively, to get (2, 5, 6) [see (2)]. By elimination, we find (1, 2, 5) and (1, 2, 6) [see (3) and (4)], respectively.

Next we show how to get the four relations involving w_1, w_2, w_3, and w_4. Suppose that

$$w_3 = B_1 w_1 + B_2 w_2$$

where B_1 and B_2 are constants to be determined. We require that $|\arg(1 - z)| < \pi$ and temporarily suppose that $R(a + b - c) < 0$ and $R(c - 1) < 0$ so that w_1 and w_2 converge at $z = 1$ and w_3 converges at $z = 0$. Let $z \to 0$ from the right. Then $w_1 = 1$, $w_2 = 0$, and B_1

follows from 3 13 1(1) Let $z \to 1$ from the left Again use 3 13 1(1) and B_2 readily follows The formula (1, 2, 3) is given by (5) The restrictions on the parameters used in its proof can be relaxed by analytic continuation, provided, of course, that we stay away from the singular points $z = 0$ and $z = 1$ The (1, 2, 3) formula may also be derived from a Mellin–Barnes contour integral, see Whittaker and Watson (1927, p 290) Replace a and b by $(c - a)$ and $(c - b)$, respectively, in (1, 2, 3) to get (1, 2 4) [see (6)] Then (1, 3, 4) and (2, 3, 4) [see (7) and (8)] follow by elimination

We have thus far obtained eight of the twenty relations The remaining twelve can be found by elimination Thus, for example, (3, 4, 5) follows from (1, 3, 4), (2, 3, 4), and (1, 2, 5) If D_1, D_2, and D_3 represent the domains $|z| < 1$, $|z - 1| < 1$, and $|z| > 1$, respectively, then (1)–(4) and (5)–(8) give the relations connecting the functions in D_1 and D_3, and D_1 and D_2, respectively The formulas (9)–(12) serve for analytic continuation between D_2 and D_3 There remain eight expressions which involve series that converge in a different domain These are given by (13)–(16) In (1)–(4) and (13)–(16), $0 < \arg z < 2\pi$ In (5)–(9), $|\arg(1 - z)| < \pi$ The right hand sides of (11), (12) are given with the wrong sign in Erdelyi *et al* [1953, p 107, Eqs (38), (40)]

$$w_1 = \frac{\Gamma(b-a)\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-a)}\,w_5 + \frac{\Gamma(a-b)\,\Gamma(c)}{\Gamma(a)\,\Gamma(c-b)}\,w_6 \tag{1}$$

$$w_2 = \frac{\Gamma(b-a)\,\Gamma(2-c)\,e^{i\pi(1-c)}}{\Gamma(1-a)\,\Gamma(1+b-c)}\,w_5 + \frac{\Gamma(a-b)\,\Gamma(2-c)\,e^{i\pi(1-c)}}{\Gamma(1-b)\,\Gamma(1+a-c)}\,w_6 \tag{2}$$

$$w_5 = \frac{\Gamma(1+a-b)\,\Gamma(1-c)}{\Gamma(1-b)\,\Gamma(1+a-c)}\,w_1 + \frac{\Gamma(1+a-b)\,\Gamma(c-1)\,e^{i\pi(c-1)}}{\Gamma(a)\,\Gamma(c-b)}\,w_2 \tag{3}$$

$$w_6 = \frac{\Gamma(1+b-a)\,\Gamma(1-c)}{\Gamma(1-a)\,\Gamma(1+b-c)}\,w_1 + \frac{\Gamma(1+b-a)\,\Gamma(c-1)\,e^{i\pi(c-1)}}{\Gamma(b)\,\Gamma(c-a)}\,w_2 \tag{4}$$

$$w_3 = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(b+1-c)\,\Gamma(a+1-c)}\,w_1 + \frac{\Gamma(a+b+1-c)\,\Gamma(c-1)}{\Gamma(a)\,\Gamma(b)}\,w_2 \tag{5}$$

$$w_4 = \frac{\Gamma(c+1-a-b)\,\Gamma(1-c)}{\Gamma(1-a)\,\Gamma(1-b)}\,w_1 + \frac{\Gamma(c+1-a-b)\,\Gamma(c-1)}{\Gamma(c-a)\,\Gamma(c-b)}\,w_2 \tag{6}$$

$$w_1 = \frac{\Gamma(c-a-b)\,\Gamma(c)}{\Gamma(c-a)\,\Gamma(c-b)}\,w_3 + \frac{\Gamma(a+b-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\,w_4 \tag{7}$$

$$w_2 = \frac{\Gamma(c-a-b)\,\Gamma(2-c)}{\Gamma(1-a)\,\Gamma(1-b)}\,w_3 + \frac{\Gamma(a+b-c)\,\Gamma(2-c)}{\Gamma(a+1-c)\,\Gamma(b+1-c)}\,w_4 \tag{8}$$

$$w_3 = \frac{\Gamma(b-a)\,\Gamma(a+b+1-c)\,e^{-i\pi a}}{\Gamma(1+b-c)\,\Gamma(b)}\,w_5 + \frac{\Gamma(a-b)\,\Gamma(a+b+1-c)\,e^{-i\pi b}}{\Gamma(1+a-c)\,\Gamma(a)}\,w_6\,. \tag{9}$$

$$w_4 = \frac{\Gamma(b-a)\,\Gamma(c+1-a-b)\,e^{i\pi(b-c)}}{\Gamma(1-a)\,\Gamma(c-a)}\,w_5 + \frac{\Gamma(a-b)\,\Gamma(c+1-a-b)\,e^{i\pi(a-c)}}{\Gamma(1-b)\,\Gamma(c-b)}\,w_6\,. \tag{10}$$

$$w_5 = \frac{\Gamma(c-a-b)\,\Gamma(1+a-b)\,e^{i\pi a}}{\Gamma(1-b)\,\Gamma(c-b)}\,w_3 + \frac{\Gamma(a+b-c)\,\Gamma(1+a-b)\,e^{i\pi(c-b)}}{\Gamma(1+a-c)\,\Gamma(a)}\,w_4\,. \tag{11}$$

$$w_6 = \frac{\Gamma(c-a-b)\,\Gamma(1+b-a)\,e^{i\pi b}}{\Gamma(1-a)\,\Gamma(c-a)}\,w_3 + \frac{\Gamma(a+b-c)\,\Gamma(1+b-a)\,e^{i\pi(c-a)}}{\Gamma(1+b-c)\,\Gamma(b)}\,w_4\,. \tag{12}$$

$$\frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\,w_1 = \frac{\Gamma(b)\,\Gamma(1+a-c)\,e^{i\pi b}}{\Gamma(a+b+1-c)}\,w_3 - \frac{\Gamma(c-b)\,\Gamma(1+a-c)\exp[i\pi(1+b-c)]}{\Gamma(1+a-b)}\,w_5; \tag{13}$$

interchange a and b to get the relation connecting w_1, w_3, and w_6.

$$\frac{\Gamma(1+b-c)\,\Gamma(1-b)}{\Gamma(2-c)}\,w_2 = \frac{\Gamma(1+b-c)\,\Gamma(a)\exp[i\pi(1+b-c)]}{\Gamma(a+b+1-c)}\,w_3 - \frac{\Gamma(a)\,\Gamma(1-b)\exp[i\pi(1+b-c)]}{\Gamma(1+a-b)}\,w_5; \tag{14}$$

interchange a and b to get the relation connecting w_2, w_3, and w_6.

$$\frac{\Gamma(c-a)\,\Gamma(a)}{\Gamma(c)}\,w_1 = \frac{\Gamma(c-a)\,\Gamma(1-b)\exp[i\pi(c-a)]}{\Gamma(1+c-a-b)}\,w_4 - \frac{\Gamma(a)\,\Gamma(1-b)\exp[i\pi(1-a)]}{\Gamma(1+a-b)}\,w_5; \tag{15}$$

interchange a and b to get the relation between w_1, w_4, and w_6.

$$\frac{\Gamma(1-a)\,\Gamma(1+a-c)}{\Gamma(2-c)}\,w_2 = \frac{\Gamma(c-b)\,\Gamma(1-a)\exp[i\pi(1-a)]}{\Gamma(1+c-a-b)}\,w_4 - \frac{\Gamma(1+a-c)\,\Gamma(c-b)\exp[i\pi(1-a)]}{\Gamma(1+a-b)}\,w_5; \tag{16}$$

interchange a and b to get the relation between w_2, w_4, and w_6.

3.10 The Complete Solution

We now take up the question of obtaining the complete solution to the hypergeometric equation 3 7(1) in the singular cases previously mentioned Recall that w_1 and w_2 are not independent or one of these is not defined if c is an integer or zero, and none of the numbers $a, b, c-a, c-b$ is an integer The same may be said for the pairs w_3, w_4 and w_5, w_6 if $(a+b+1-c)$ and $(a+1-b)$, respectively, are integers or zero In the degenerate cases where any of the numbers $a, b, c-a, c-b$ is an integer, fundamental solutions are easily picked out from Kummer's set of 24 solutions, see the concluding remarks of 3 8 In this section we develop expansions which are solutions of 3 7(1) when the denominator parameters of the ${}_2F_1$'s appearing in $w_1, w_2, \ldots, w_6$ are integers Thus, for example, we develop expansions which satisfy 3 7(1) and which are independent of w_1 when c is a positive integer or zero Furthermore, the analytic continuation of these solutions is automatically provided. Except in some of the degenerate cases, the solutions involve logarithms and are called logarithmic solutions In this connection, the degenerate solutions are a simple by-product of the logarithmic solutions A table listing the degenerate solutions and the logarithmic solutions for various choices of the parameters a, b, and c is given in Erdelyi *et al* (1953, Vol 1, pp 71–73) We have made several improvements on this table which we present at the end of this section In this connection, see also Norlund (1963)

To simplify the analysis, consider

$$
\begin{aligned}
V(z) &= (ze^{-i\pi})^{-a_1}\,{}_2F_1\left(\begin{matrix}1+a_1-b_0,\,1+a_1-b_1\\ 1+a_1-a_2\end{matrix}\middle|\,z^{-1}\right)\\
&= \frac{\Gamma(b_0-b_1)\,\Gamma(1+a_1-a_2)}{\Gamma(b_0-a_2)\,\Gamma(1+a_1-b_1)}(ze^{-i\pi})^{(1-b_0)}\,{}_2F_1\left(\begin{matrix}1+a_1-b_0,\,1+a_2-b_0\\ 1+b_1-b_0\end{matrix}\middle|\,z\right)\\
&\quad+\frac{\Gamma(b_1-b_0)\,\Gamma(1+a_1-a_2)}{\Gamma(b_1-a_2)\,\Gamma(1+a_1-b_0)}(ze^{-i\pi})^{(1-b_1)}\,{}_2F_1\left(\begin{matrix}1+a_1-b_1,\,1+a_2-b_1\\ 1+b_0-b_1\end{matrix}\middle|\,z\right),\\
&\quad 0<\arg z<2\pi
\end{aligned}
\tag{1}
$$

This is advantageous as it can represent any of the formulas 3 9(1–4) by making the substitutions in the accompanying tabulation

$V(z)$	Equation	z	a_1	a_2	b_0	b_1
w_1	3 9(1)	$z^{-1}e^{2\pi i}$	0	$1-c$	$1-a$	$1-b$
$e^{i\pi(c-1)}w_2$	3 9(2)	$z^{-1}e^{2\pi i}$	$1-c$	0	$1-a$	$1-b$
w_5	3 9(3)	z	a	b	1	c
w_6	3 9(4)	z	b	a	1	c

(2)

Similarly Eqs. 3.9(5–8) are given by

$$
\begin{aligned}
W(z) &= {}_2F_1\left(\begin{matrix}1+a_1-b_0\,,\,b_1-a_2\\ 1+a_1-a_2\end{matrix}\,\middle|\,z\right)\\
&= \frac{\Gamma(b_0-b_1)\,\Gamma(1+a_1-a_2)}{\Gamma(b_0-a_2)\,\Gamma(1+a_1-b_1)}\,{}_2F_1\left(\begin{matrix}1+a_1-b_0\,,\,b_1-a_2\\ 1+b_1-b_0\end{matrix}\,\middle|\,1-z\right),\\
&\quad+(1-z)^{b_0-b_1}\frac{\Gamma(b_1-b_0)\,\Gamma(1+a_1-a_2)}{\Gamma(b_1-a_2)\,\Gamma(1+a_1-b_0)}\,{}_2F_1\left(\begin{matrix}1+a_1-b_1\,,\,b_0-a_2\\ 1+b_0-b_1\end{matrix}\,\middle|\,1-z\right),\\
&\quad|\arg(1-z)|<\pi.
\end{aligned}
\tag{3}
$$

(4)

$W(z)$	Equation	z	a_1	a_2	b_0	b_1
w_3	3.9(5)	$1-z$	a	$c-b$	1	c
$(1-z)^{a+b-c}w_4$	3.9(6)	$1-z$	$c-a$	b	1	c
w_1	3.9(7)	z	a	$1+a-c$	1	$a+b+1-c$
$z^{c-1}w_2$	3.9(8)	z	$a+1-c$	a	1	$a+b+1-c$

To represent 3.9(9–12), it is convenient to use two formulas:

$$
\begin{aligned}
T(z) &= {}_2F_1\left(\begin{matrix}1+a_1-b_0\,,\,1+a_1-b_1\\ 1+a_1-a_2\end{matrix}\,\middle|\,1-z\right)\\
&= \exp[i\pi(b_0-1-a_1)]\,\frac{\Gamma(b_0-b_1)\,\Gamma(1+a_1-a_2)}{\Gamma(b_0-a_2)\,\Gamma(1+a_1-b_1)}\,(z^{-1}e^{i\pi})^{1+a_1-b_0}\\
&\quad\times{}_2F_1\left(\begin{matrix}1+a_1-b_0\,,\,b_1-a_2\\ 1+b_1-b_0\end{matrix}\,\middle|\,z^{-1}\right)\\
&\quad+\exp[i\pi(b_1-1-a_1)]\,\frac{\Gamma(b_1-b_0)\,\Gamma(1+a_1-a_2)}{\Gamma(b_1-a_2)\,\Gamma(1+a_1-b_0)}\,(z^{-1}e^{i\pi})^{1+a_1-b_1}\\
&\quad\times{}_2F_1\left(\begin{matrix}1+a_1-b_1\,,\,b_0-a_2\\ 1+b_0-b_1\end{matrix}\,\middle|\,z^{-1}\right),\qquad 0<\arg z<2\pi.
\end{aligned}
\tag{5}
$$

(6)

$T(z)$	Equation	a_1	a_2	b_0	b_1
w_3	3.9(9)	a	$c-b$	1	$1+a-b$
$(1-z)^{a+b-c}w_4$	3.9(10)	$c-a$	b	1	$1+b-a$

Note that

$$z^{1+a_1-b_0}T(z) = W(1-z^{-1}). \tag{7}$$

$$U(z) = (z^{-1}e^{i\pi})^{1+a_1-b_0}\,{}_2F_1\left(\begin{matrix}1+a_1-b_0,\,b_1-a_2\\ 1+a_1-a_2\end{matrix}\middle|\,z^{-1}\right)$$

$$= \frac{\Gamma(b_0-b_1)\,\Gamma(1+a_1-a_2)}{\Gamma(b_0-a_2)\,\Gamma(1+a_1-b_1)}\exp[i\pi(1+a_1-b_0)]$$

$$\times\,{}_2F_1\left(\begin{matrix}1+a_1-b_0,\,1+a_2-b_0\\ 1+b_1-b_0\end{matrix}\middle|\,1-z\right)$$

$$+\frac{\Gamma(b_1-b_0)\,\Gamma(1+a_1-a_2)}{\Gamma(b_1-a_2)\,\Gamma(1+a_1-b_0)}\exp[i\pi(1+a_1-b_1)](1-z)^{b_0-b_1}$$

$$\times\,{}_2F_1\left(\begin{matrix}1+a_1-b_1,\,1+a_2-b_1\\ 1+b_0-b_1\end{matrix}\middle|\,1-z\right),\qquad 0<\arg z<2\pi \tag{8}$$

$U(z)$	Equation	a_1	a_2	b_0	b_1
w_5	3.9(11)	a	b	1	$1+a+b-c$
w_6	3.9(12)	b	a	1	$1+a+b-c$

(9)

Note that

$$\exp(-i\pi a_1)(1-z)^{1-b_0}U(z) = V((1-z)\,e^{2\pi i}) \tag{10}$$

In $V(z)$, suppose that $b_1 - b_0 = s + \epsilon$ where s is a positive integer or zero. We further suppose that neither of the numbers $1 + a_1 - b_0$ nor $1 + a_2 - b_0$ is a negative integer or zero. Then

$$V(z) = \frac{\Gamma(-s-\epsilon)\,\Gamma(1+a_1-a_2)}{\Gamma(b_0-a_2)\,\Gamma(1+a_1-b_1)}(ze^{-i\pi})^{(1-b_0)}\,{}_2F_1\left(\begin{matrix}1+a_1-b_0,\,1+a_2-b_0\\ 1+s+\epsilon\end{matrix}\middle|\,z\right)$$

$$+\frac{\Gamma(s+\epsilon)\,\Gamma(1+a_1-a_2)}{\Gamma(b_1-a_2)\,\Gamma(1+a_1-b_0)}(ze^{-i\pi})^{(1-b_1)}\,{}_2F_1\left(\begin{matrix}1+a_1-b_1,\,1+a_2-b_1\\ 1-s-\epsilon\end{matrix}\middle|\,z\right) \tag{11}$$

Now the second ${}_2F_1$ can be expressed as two sums, viz., $\sum_{k=0}^{s-1}$ and $\sum_{k=s}^{\infty}$. In the first sum we can let $\epsilon \to 0$. The second sum can be expressed in the form $\sum_{k=0}^{\infty}$, and with

$$V_1(z) = \frac{(ze^{-i\pi})^{(1-b_1)}\Gamma(1+a_1-a_2)}{\Gamma(b_1-a_2)\,\Gamma(1+a_1-b_0)}$$

$$\times\sum_{k=0}^{s-1}\frac{(1+a_1-b_1)_k(1+a_2-b_1)_k(s-1-k)!\,(-)^k z^k}{k!}$$

$$= \frac{(z^{-1}e^{i\pi})^{b_0}\Gamma(1+a_1-a_2)(1+a_1-b_1)_s\,{}_1(1+a_2-b_1)_{s-1}}{\Gamma(b_1-a_2)\,\Gamma(1+a_1-b_0)(s-1)!}$$

$$\times\,{}_3F_2\left(\begin{matrix}1-s,\,1,\,1\\ 1+b_0-a_1,\,1+b_0-a_2\end{matrix}\middle|\,z^{-1}\right), \tag{12}$$

we can write

$$V(z) - V_1(z) = \frac{(-)^{s+1}\Gamma(1-\epsilon)\,\Gamma(1+a_1-a_2)(ze^{-i\pi})^{(1-b_0)}}{\epsilon(1+\epsilon)_s\Gamma(b_0-a_2)\,\Gamma(1+a_1-b_1)}$$

$$\times \left\{ {}_2F_1\left(\begin{matrix}1+a_1-b_0\,,\,1+a_2-b_0\\ 1+s+\epsilon\end{matrix}\middle|\, z\right)\right.$$

$$-\frac{(ze^{-i\pi})^{-\epsilon}(1+s)_\epsilon(1+a_1-b_0)_{-\epsilon}(1+a_2-b_0)_{-\epsilon}\sin\pi(b_0-a_2+\epsilon)}{\Gamma(1-\epsilon)\sin\pi(b_0-a_2)}$$

$$\left.\times\, {}_3F_2\left(\begin{matrix}1+a_1-b_0-\epsilon,\,1+a_2-b_0+\epsilon,\,1\\ 1-\epsilon,\,1+s\end{matrix}\middle|\, z\right)\right\}.$$

Now let $\epsilon \to 0$. Then by L'Hospital's theorem

$$V(z) = V_1(z) + \frac{(-)^{s+1}(ze^{-i\pi})^{(1-b_0)}\Gamma(1+a_1-a_2)}{s!\,\Gamma(b_0-a_2)\,\Gamma(1+a_1-b_1)}$$

$$\times \left\{[\gamma + \ln(ze^{-i\pi}) + \psi(1+a_1-b_0) + \psi(b_0-a_2) - \psi(1+s)]\right.$$

$$\times\, {}_2F_1\left(\begin{matrix}1+a_1-b_0\,,\,1+a_2-b_0\\ 1+s\end{matrix}\middle|\, z\right)$$

$$+\sum_{k=0}^{\infty}\frac{(1+a_1-b_0)_k(1+a_2-b_0)_k z^k}{(1+s)_k k!}$$

$$\times\,(\psi(1+a_1-b_0+k) - \psi(1+a_1-b_0)$$

$$+\,\psi(1+a_2-b_0+k) - \psi(1+a_2-b_0)$$

$$\left.-\,\psi(1+s+k) + \psi(1+s) - \psi(1+k) + \psi(1))\right\}, \tag{13}$$

$b_1 = b_0 + s$, s is a positive integer or zero,

$1 + a_1 - b_0$ is not a negative integer or zero,

$1 + a_2 - b_0$ is not an integer or zero,

$0 < \arg z < 2\pi$.

It is understood that if $s = 0$, $V_1(z)$ is nil, i.e., $\sum_{k=0}^{s-1}$ is nil if $s = 0$. This convention is retained throughout.

We can also derive (13) from the Mellin–Barnes contour integral 3.6(28). For example, if $V(z) = w_1$, then $a = b + s$. The integrand of 3.6(28) (there replace t by u) has simple poles at $u = -b - k$, $k = 0, 1,\ldots, s-1$,

$s > 0$ If $u = -b - s - k$, $k = 0, 1,$, these are poles of the second order Now

$$A = \frac{\Gamma(a+u)\,\Gamma(b+u)\,\Gamma(-u)(-z)^u}{\Gamma(c+u)} = \frac{\pi\Gamma(a+u)\,\Gamma(-u)(-z)^u}{\sin\pi(b+u)\,\Gamma(c+u)\,\Gamma(1-b-u)},$$

and the residue of A at $u = -b - k$ is

$$\frac{d}{du}\{(u+b+k)\,A\}_{u=-b-k} = \frac{(ze^{-i\pi})^{-b}\Gamma(b+k)(1-b-c)_k(s-1-k)!\,(-z)^{-k}}{\Gamma(c-b)\,k!} \tag{14}$$

Again, we can write

$$A = \frac{(-)^s\pi^2\Gamma(-u)(-z)^u}{\sin^2\pi(a+u)\,\Gamma(1-a-u)\,\Gamma(1-b-u)\,\Gamma(c+u)},$$

and the residue of A at $u = -a - k$ is

$$\begin{aligned}\frac{d}{du}\{(u+a+k)^2A\}_{u=-a-k} = {} & \frac{(-)^{s+1}(z^{-1}e^{i\pi})^a\Gamma(a+k)(1+a-c)_k z^{-k}}{\Gamma(c-a)(s+k)!\,k!} \\ & \times [\ln(z^{-1}e^{i\pi}) + \psi(a+k) \\ & + \psi(c-a-k) - \psi(1+k) - \psi(1+s+k)]\end{aligned} \tag{15}$$

When the contour integral is evaluated as a sum of the residues of its integrand due to the poles enumerated above, (14) leads to the polynomial $V_1(z)$ and (15) leads to the series expansion for $V(z) - V_1(z)$, all of this for $V(z) = w_1$ as cited

We now consider the restrictions imposed on $1 + a_1 - b_0$ and $1 + a_2 - b_0$ in (13) If $1 + a_1 - b_0$ is a negative integer or zero, then from (12), $V_1(z) = 0$ and with the aid of 2 4(13),

$$V(z) = \frac{(-)^m(m+s)!\,\Gamma(1+a_1-a_2)}{s!\,\Gamma(b_0-a_2)}(ze^{-i\pi})^{1-b_0}\,{}_2F_1\left(\begin{matrix} m & 1+a_2-b_0 \\ & 1+s \end{matrix}\middle| z\right)$$

$$1 + a_1 - b_0 = -m, \qquad m \text{ a positive integer or zero,} \tag{16}$$

which is the statement (1)

If $(b_0 - a_2)$ is a negative integer or zero, a similar analysis gives

$$\begin{aligned}V(z) = V_1(z) + {} & \frac{(-)^{m+s+1}(m-1)!\,\Gamma(1+a_1-a_2)}{s!\,\Gamma(1+a_1-b_1)}(ze^{-i\pi})^{(1-b_0)} \\ & \times {}_2F_1\left(\begin{matrix} m & 1+a_1-b_0 \\ & 1+s \end{matrix}\middle| z\right),\end{aligned} \tag{17}$$

$$V_1(z) = \frac{(s-1)!\,\Gamma(1+a_1-a_2)(ze^{-i\pi})^{(1-b_1)}}{(s-m)!\,\Gamma(1+a_1-b_0)}$$
$$\times \sum_{k=0}^{s-m} \frac{(m-s)_k(1+a_1-b_1)_k z^k}{(1-s)_k k!}$$
$$= \frac{(m-1)!\,\Gamma(1+a_1-a_2)\,\Gamma(a_1-a_2)}{(s-m)!\,\Gamma(1+a_1-b_0)\,\Gamma(1+a_1-b_1)}(ze^{-i\pi})^{(1-b_1)}z^{s-m}$$
$$\times {}_2F_1\left(\begin{matrix} m-s, m \\ m+b_0-a_1 \end{matrix}\middle| z^{-1}\right),$$
$$b_0 - a_2 = 1 - m, \qquad m \text{ a positive integer,}$$

and this too could be deduced directly from (1). In (17), the second form for $V_1(z)$ follows from 3.2(7). Note that $V_1(z)$ vanishes for $m > s$.

Now suppose that $(b_0 - a_2)$ is a positive integer. Then application of 2.4(14) to (13) gives

$$V(z) = V_1(z) + \frac{(-)^{s-1}(ze^{-i\pi})^{(1-b_0)}\Gamma(1+a_1-a_2)}{s!\,m!\,\Gamma(1+a_1-b_1)}$$
$$\times \Big\{[\gamma + \ln(ze^{-i\pi}) + \psi(1+a_1-b_0)$$
$$+ \psi(1+m) - \psi(1+s)]\,{}_2F_1\left(\begin{matrix} -m, 1+a_1-b_0 \\ 1+s \end{matrix}\middle| z\right)$$
$$+ \sum_{k=0}^{m} \frac{(-m)_k(1+a_1-b_0)_k z^k}{(1+s)_k k!}$$
$$\times (\psi(1+a_1-b_0+k) - \psi(1+a_1-b_0) + \psi(1+m-k) - \psi(1+m)$$
$$- \psi(1+s+k) + \psi(1+s) - \psi(1+k) + \psi(1)) + (-)^m m!$$
$$\times \sum_{k=m+1}^{\infty} \frac{(1+a_1-b_0)_k(k-m-1)!\,z^k}{(1+s)_k k!}\Big\},$$
$$b_1 = b_0 + s, \qquad 1 + a_2 - b_0 = -m, \qquad 0 < \arg z < 2\pi, \tag{18}$$

where m and s are positive integers or zero. Note that the last infinite series in (18) can be expressed in terms of a ${}_3F_2$.

Thus (13) is valid without the restrictions on $(1 + a_1 - b_0)$ and $(1 + a_2 - b_0)$. The point is that without these conditions, the expansion is free of logarithms unless $(b_0 - a_2)$ is a positive integer.

If $a_1 = b_0 + \nu$, $a_2 = b_0 - \mu - \lambda - 1$, and $b_1 = b_0 - \mu$, or if $a_1 = b_0 + \nu + \mu$, $a_2 = b_0 - \lambda - 1$, and $b_1 = b_0 + \mu$, where μ, ν, and λ are positive integers or zero, then the logarithmic solution for $V(z)$ can be put in the form 3.4(15). If $b_0 = a_2$ or $b_1 = a_2$, see 3.7(16).

For $W(z)$ we have

$$W_1(z) = \frac{(1-z)^{-s}\,\Gamma(1+a_1-a_2)}{\Gamma(b_1-a_2)\,\Gamma(1+a_1-b_0)}$$
$$\times \sum_{k=0}^{s-1} \frac{(1+a_1-b_1)_k(b_0-a_2)_k(s-1-k)!\,(-)^k(1-z)^k}{k!}$$
$$= \frac{(-)^{s+1}\Gamma(1+a_1-a_2)(1+a_1-b_1)_{s-1}(b_0-a_2)_{s-1}}{(1-z)\Gamma(b_1-a_2)\Gamma(1+a_1-b_0)(s-1)!}$$
$$\times {}_3F_2\left(\begin{matrix}1-s, 1, 1\\ 1+b_0-a_1, 2-b_1+a_2\end{matrix}\middle|\frac{1}{1-z}\right), \tag{19}$$

$$W(z) = W_1(z) + \frac{(-)^{s+1}\Gamma(1+a_1-a_2)}{s!\,\Gamma(b_0-a_2)\,\Gamma(1+a_1-b_1)}$$
$$\times \Big\{[\gamma + \ln(1-z) + \psi(1+a_1-b_0) + \psi(b_1-a_2) - \psi(1+s)]$$
$$\times {}_2F_1\left(\begin{matrix}1+a_1-b_0, b_1-a_2\\ 1+s\end{matrix}\middle|1-z\right)$$
$$+ \sum_{k=0}^{\infty} \frac{(1+a_1-b_0)_k(b_1-a_2)_k(1-z)^k}{(1+s)_k k!}$$
$$\times (\psi(1+a_1-b_0+k) - \psi(1+a_1-b_0) + \psi(b_1-a_2+k) - \psi(b_1-a_2)$$
$$- \psi(1+s+k) + \psi(1+s) - \psi(1+k) + \psi(1))\Big\}, \tag{20}$$

where $b_1 = b_0 + s$, s is a positive integer or zero, neither of the numbers $1 + a_1 - b_0$ nor $b_1 - a_2$ is a negative integer or zero, and $|\arg(1-z)| < \pi$

If $(1 + a_1 - b_0)$ is a negative integer or zero, then directly from (3), and without the restriction that b_1 is a positive integer, we have

$$W(z) = {}_2F_1\left(\begin{matrix}-m, b_1-a_2\\ 1-m-a_2\end{matrix}\middle|z\right) = \frac{(b_1)_m}{(a_2)_m}\,{}_2F_1\left(\begin{matrix}-m, b_1-a_2\\ b_1\end{matrix}\middle|1-z\right),$$
$$a_1 = -m, \qquad m \text{ a positive integer or zero} \tag{21}$$

If $(b_1 - a_2)$ is a negative integer or zero, we again arrive at the form (21) since $W(z)$ is symmetric in its numerator parameters

Suppose now that $b_0 = b_1 + s$, s a positive integer or zero Then $W(z)$ is given by the right-hand side of (20) if there we interchange the roles of b_0 and b_1 and multiply throughout by $(1-z)^s$

If $a_1 = \nu + 1$, $a_2 = -\mu - \lambda$, and $b_1 = \nu - \lambda + 1$, or if $a_1 = \nu + \mu + 1$, $a_2 = -\lambda$, and $b_1 = \nu - \lambda + 1$, where μ, ν, and λ are positive integers or zero, then the logarithmic solution is also given by 3 4(15) If $b_0 = a_2$ or $b_1 = 1 + a_1$, see 3 7(16)

Similarly, from (5)

$$T_1(z) = \frac{\exp[i\pi(b_1 - 1 - a_1)]\,\Gamma(1 + a_1 - a_2)(z^{-1}e^{i\pi})^{1+a_1-b_1}}{\Gamma(b_1 - a_2)\,\Gamma(1 + a_1 - b_0)}$$
$$\times \sum_{k=0}^{s-1} \frac{(1 + a_1 - b_1)_k(b_0 - a_2)_k(s - 1 - k)!\,(-)^k z^{-k}}{k!}$$
$$= \frac{(-)^{s+1}z^{b_0-a_1}\Gamma(1 + a_1 - a_2)(1 + a_1 - b_1)_{s-1}(b_0 - a_2)_{s-1}}{\Gamma(b_1 - a_2)\,\Gamma(1 + a_1 - b_0)(s - 1)!}$$
$$\times {}_3F_2\left(\begin{matrix} 1 - s, 1, 1 \\ 1 + b_0 - a_1, 2 + a_2 - b_0 \end{matrix}\middle| z\right), \tag{22}$$

$$T(z) = T_1(z) + \frac{(-)^{s+1}\Gamma(1 + a_1 - a_2)\,z^{b_0-a_1-1}}{s!\,\Gamma(b_0 - a_2)\,\Gamma(1 + a_1 - b_1)}$$
$$\times \Big\{[\gamma - \ln z + \psi(1 + a_1 - b_0) + \psi(b_1 - a_2) - \psi(1 + s)]$$
$$\times {}_2F_1\left(\begin{matrix} 1 + a_1 - b_0, b_1 - a_2 \\ 1 + s \end{matrix}\middle| z^{-1}\right) + \sum_{k=0}^{\infty} \frac{(1 + a_1 - b_0)_k(b_1 - a_2)_k z^{-k}}{(1 + s)_k k!}$$
$$\times (\psi(1 + a_1 - b_0 + k) - \psi(1 + a_1 - b_0)$$
$$+ \psi(b_1 - a_2 + k) - \psi(b_1 - a_2) - \psi(1 + s + k)$$
$$+ \psi(1 + s) - \psi(1 + k) + \psi(1))\Big\}, \tag{23}$$

where $b_1 = b_0 + s$, s is a positive integer or zero, neither of the numbers $1 + a_1 - b_0$ nor $b_1 - a_2$ is a negative integer or zero, and $0 < \arg z < 2\pi$. The restriction on the numbers $1 + a_1 - b_0$ and $b_1 - a_2$ is not essential. In the absence of this restriction, $T(z)$ is not logarithmic and the forms for $T(z)$ can be deduced directly from (5). The discussion is like that for $W(z)$ in view of (7).

If $a_1 = \nu + 1$, $a_2 = -\mu - \lambda$, and $b_1 = -\mu + 1$ or if $a_1 = \nu + \mu + 1$, $a_2 = -\lambda$, and $b_1 = \mu + 1$, where μ, ν, and λ are positive integers or zero, then the logarithmic solution for $T(z)$ follows from 3.4(15). If $b_0 = a_2$ or $b_1 = a_2$, see 3.7(16).

For $U(z)$ [see (8)] we have

$$U_1(z) = \frac{\Gamma(1 + a_1 - a_2)\exp[i\pi(1 + a_1 - b_1)](1 - z)^{-s}}{\Gamma(b_1 - a_2)\,\Gamma(1 + a_1 - b_0)}$$
$$\times \sum_{k=0}^{s-1} \frac{(1 + a_1 - b_1)_k(1 + a_2 - b_1)_k(s - 1 - k)!\,(-)^k(1 - z)^k}{k!}$$
$$= \frac{\exp[i\pi(a_1 - b_0)]\,\Gamma(1 + a_1 - a_2)(1 + a_1 - b_1)_{s-1}(1 + a_2 - b_1)_{s-1}}{(1 - z)\,\Gamma(b_1 - a_2)\,\Gamma(1 + a_1 - b_0)(s - 1)!}$$
$$\times {}_3F_2\left(\begin{matrix} 1 - s, 1, 1 \\ 1 + b_0 - a_1, 1 + b_0 - a_2 \end{matrix}\middle| \frac{1}{1 - z}\right), \tag{24}$$

$$
\begin{aligned}
U(z) = U_1(z) &+ \frac{(-)^{s+1}\Gamma(1+a_1-a_2)\exp[i\pi(1+a_1-b_0)]}{s!\,\Gamma(1+a_1-b_1)\,\Gamma(b_0-a_2)} \\
&\times \Big\{[\gamma + \ln(1-z)\,e^{i\pi} + \psi(1+a_1-b_0) + \psi(b_0-a_2) - \psi(1+s)] \\
&\times {}_2F_1\left(\begin{matrix}1+a_1-b_0,\ 1+a_2-b_0\\ 1+s\end{matrix}\middle|\, 1-z\right) \\
&+ \sum_{k=0}^{\infty} \frac{(1+a_1-b_0)_k(1+a_2-b_0)_k(1-z)^k}{(1+s)_k k!} \\
&\times (\psi(1+a_1-b_0+k) - \psi(1+a_1-b_0) + \psi(1+a_2-b_0+k) \\
&- \psi(1+a_2-b_0) - \psi(1+s+k) + \psi(1+s) - \psi(1+k) + \psi(1))\Big\},
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
b_1 = b_0 + s,\quad & s \text{ a positive integer or zero,} \\
& 1 + a_1 - b_0 \text{ is not a negative integer or zero,} \\
& 1 + a_2 - b_0 \text{ is not an integer or zero,} \\
& 0 < \arg z < 2\pi
\end{aligned}
$$

If $b_0 = b_1 + s$, s as in (25), then $U(z)$ is given by the right-hand side of (25) if there we interchange the roles of b_0 and b_1 and multiply throughout by $(1-z)^s$

Concerning the restrictions in (25), if $(1 + a_1 - b_0)$ is a negative integer or zero, or if $(b_0 - a_2)$ is a negative integer or zero, then $U(z)$ is not logarithmic, and the forms for $U(z)$ can be inferred directly from (8) The discussion is akin to that for $V(z)$ in view of (10) Also

$$
\begin{aligned}
U(z) = U_1(z) &+ \frac{(-)^{s+1}\Gamma(1+a_1-a_2)\exp[i\pi(1+a_1-b_0)]}{s!\,m!\,\Gamma(1+a_1-b_1)} \\
&\times \Big\{[\gamma + \ln(1-z)\,e^{i\pi} + \psi(1+a_1-b_0) + \psi(1+m) - \psi(1+s)] \\
&\times {}_2F_1\left(\begin{matrix}-m,\ 1+a_1-b_0\\ 1+s\end{matrix}\middle|\, 1-z\right) + \sum_{k=0}^{m} \frac{(-m)_k(1+a_1-b_0)_k(1-z)^k}{(1+s)_k k!} \\
&\times (\psi(1+a_1-b_0+k) - \psi(1+a_1-b_0) + \psi(1+m-k) \\
&- \psi(1+m) - \psi(1+s+k) - \psi(1+k) + \psi(1)) \\
&+ (-)^m m! \sum_{k=m+1}^{\infty} \frac{(k-m-1)!\,(1+a_1-b_0)_k(1-z)^k}{(1+s)_k k!}\Big\}
\end{aligned}
\tag{26}
$$

$$
b_1 = b_0 + s, \qquad 1 + a_2 - b_0 = -m, \qquad 0 < \arg z < 2\pi,
$$

where m and s are positive integers or zero

TABLE 3.2

TABLE OF SOLUTIONS IN THE DEGENERATE CASE[a]

Case	a	b	c	$c-a-b$	Degenerate solution	Second solution	Logarithmic solution
1	$-m$	n.i.	n.i.	n.i.	$w_{1,1}$	$w_{2,2}$	—
2	$m+1$	n.i.	n.i.	n.i.	$w_{2,2}$	$w_{1,1}$	—
3	$c+m$	n.i.	n.i.	n.i.	$w_{1,2}$	$w_{2,1}$	—
4	$c-m-1$	n.i.	n.i.	n.i.	$w_{2,1}$	$w_{1,2}$	—
5	$-m$	n.i.	n.i.	s	$w_{1,1}$	$w_{2,2}$	$w_{2,2} = z^{1-c}(1-z)^s W(z)$, see (3) and (20), with $b_0 = 1, b_1 = s+1, a_1 = m+1, 1+a_1-a_2 = 2-c$
6	$m+1$	n.i.	n.i.	s	$w_{2,2}$	$w_{1,1}$	$w_{1,1} = W(z)$, see (3) and (20) with b_1 and b_0 interchanged, and then put $b_0 = 1$, $b_1 = s+1$, $1+a_1-b_1 = m+1$, $1+a_1-a_2 = c$
7	$-m$	n.i.	n.i.	$-s$	$w_{1,1}$	$w_{2,2}$	$w_{2,2} = z^{1-c}(1-z)^{-s} W(z)$, see (3) and (20) with b_1 and b_0 interchanged, and then put $b_0 = 1$, $b_1 = s+1$, $1+a_1-b_1 = m+1$, $a_2 = b$
8	$m+1$	n.i.	n.i.	$-s$	$w_{2,2}$	$w_{1,1}$	$w_{1,1} = W(z)$, see (3) and (20) with $b_0 = 1$, $b_1 = s+1$, $a_1 = m+1$, $1+a_1-a_2 = c$
9	$m+1$	$m+1+s$	n.i.	n.i.	$w_{2,2}$	$w_{1,1}$	$w_{1,1} = V(z)$, see (1) and (17) with $b_0 = -m-s$, $b_1 = -m$, $a_1 = 0$, $a_2 = 1-c$, and z replaced by $z^{-1}e^{2\pi i}$

[a] See *Notation* on p. 84 for explanation of table symbols.

TABLE 3 2 (*cont*)

Case	a	b	c	$c-a-b$	Degenerate solution	Second solution	Logarithmic solution
10	$-m$	s	n i	n i	$w_{1\,1}$	$w_{2\,2}$ deg	—
11	$-m-s$	$-m$	n i	n i	$w_{1\,1}$	$w_{2\,2}$	$w_{2\,2} = x^{1-c}(1-x)^{c-a-b}V(x)$ see (1) and (17) with $b_0 = -m-s$ $b_1 = m$ $a_1 = 0$ $a_2 = c-1$ and x replaced by $x^{-1}e^{2\pi i}$
12	$-m$	n i	$-m-s$	n i	$w_{5\,1}$	$w_{6\,2}$ deg	—
13	$-m-s$	n i	$1-s$	n i	$w_{2\,1}$	$w_{6\,2}$	$w_{6\,2} = (xe^{-\pi i})^{s}(1-x)^{c-a-b}V(x)$ see (1) and (17) with $b_0 = 1$ $b_1 = s+1$ $a_1 = 1-a$ $a_2 = 1-b$
14	$-m$	n i	$s+1$	n i	$w_{1\,1}$	$w_{6\,2}$	$w_{6\,2} = (1-x)^{c-a-b}V(x)$ see (1) and (17) with $b_0 = 1$ $b_1 = s+1$ $a_1 = m+s+1$ $1+a-a_2 = m+1+b$
15	$m+1$	n i	$1-s$	n i	$w_{2\,2}$	$w_{6\,1}$	$w_{6\,1} = (xe^{-\pi i})^{s}V(x)$ see (1) and (17) with $b_0 = 1$ $b_1 = s+1$ $a_1 = m+s+1$ $1+a_1-a_2 = m+2-b$
16	$m+s+1$	n i	$s+1$	n i	$w_{1\,2}$	$w_{6\,1}$	$w_{6\,1} = V(x)$ see (1) and (17) with $b_0 = 1$ $b_1 = s+1$ $a_1 = m+s+1$ $a_2 = b$
17	$m+1$	n i	$m+s+2$	n i	$w_{6\,1}$	$w_{4\,2}$ deg	—
18	$m+s+1$	$m+s+n+1$	$s+1$	$-2m-n-s-1$	$w_{1\,2}$	$w_{2\,1}$	$w_{2\,1} = A(x)$ see 3 4(13) with $\nu = m+s$ $\mu = n$ $\lambda = m$ and x replaced by $1-x$

19	$m+1$	$m+s+n+2$	$m+n+2$	$-m-s-1$	$w_{1,2}$	$w_{6,2}$ deg.	—
20	$m+1$	$m+s+1$	$m+n+s+2$	$n-m$	$w_{6,1}$	$w_{1,1}$	$w_{1,1} = A(z)$, see 3.4(15) with $\nu = m$, $\mu = s$, and $\lambda = n$
21	$-m$	$n+s+1$	$s+1$	$m-n$	$w_{1,1}$	$w_{6,2}$	$w_{6,2} = (z^{-1}e^{i\pi})^{m+s+1}(1-z)^{m-n}A(z)$ where $A(z)$ is given by 3.4(15) with $\nu = m$, $\mu = s$, $\lambda = n$, and z replaced by z^{-1}
22	$-m$	$n+1$	$n+s+2$	$m+s+1$	$w_{1,1}$	$w_{6,1}$ deg.	—
23	$-m-n$	$-m$	$s+1$	$2m+n+s+1$	$w_{1,1}$	$w_{4,1}$	$w_{4,1} = (1-z)^{2m+n+s+1}A(z)$ where $A(z)$ is given by 3.4(15) with $\nu = m+s$, $\mu = n$, $\lambda = m$, and z replaced by $1-z$
24	$-m-n$	$-m$	$-m-n-s$	$m-s$	$w_{6,1}$	$w_{2,2}$	$w_{2,2} = z^{m+n+s+1}(1-z)^{m-s}A(z)$ where $A(z)$ is given by 3.4(15) with $\nu = m$, $\mu = n$, and $\lambda = s$
25	$-m-n-s$	$-n-s$	$1-s$	$m+2n+s+1$	$w_{2,1}$	$w_{4,2}$	$w_{4,2} = z^{s}(1-z)^{m+2n+s+1}A(z)$ where $A(z)$ is given by 3.4(15) with $\nu = n+s$, $\mu = m$, $\lambda = n$, and z replaced by $1-z$
26	$-m-n-s-2$	$-n$	$-n-s-1$	$m+n+1$	$w_{2,1}$	$w_{6,1}$ deg.	—
27	$-m-s-1$	$n+1$	$-s$	$m-n$	$w_{2,1}$	$w_{6,1}$	$w_{6,1} = (z^{-1}e^{i\pi})^{n+1}A(z)$ where $A(z)$ is given by 3.4(15) with $\nu = n$, $\mu = s+1$, $\lambda = m$, and z replaced by z^{-1}
28	$-m$	$n+1$	$-m-s-1$	$-n-s-2$	$w_{2,2}$	$w_{3,1}$ deg.	—
29	$m+1$	$m+n+1$	$-s$	$-2m-n-s-2$	$w_{2,2}$	$w_{3,2}$	$w_{3,2} = z^{s+1}A(z)$ where $A(z)$ is given by 3.4(15) with $\nu = m+s+1$, $\mu = n$, $\lambda = m$, and z replaced by $1-z$

If $a_1 = \nu + 1$, $a_2 = -\mu - \lambda$, and $b_1 = \nu - \lambda + 1$ or if $a_1 = \nu + \mu + 1$, $a_2 = -\lambda$, and $b_1 = \nu - \lambda + 1$, where μ, ν, and λ are positive integers or zero, then the logarithmic solution for $U(z)$ can be expressed by the form 3 4(15) If $b_0 = a_2$ or $b_1 = 1 + a_1$, see 3 7(16)

Notation for the Table of Solutions in the Degenerate Case (Table 3 2)

It is convenient to introduce some notation

m, n, s	Denotes nonnegative integers
n i	Means that the quantity is not an integer
$w_{i,j}$ [see 3 8(1–24)]	As previously noted we have the 24 series of Kummer These may be divided into six sets, each set consisting of four equations We have already called the sets w_i, $i = 1, 2, \ldots, 6$ We now use $w_{i,j}$ to designate the jth equation in the ith set Thus $w_{1,1}$, $w_{2,2}$, and $w_{6,2}$ stand for Eqs 3 8(1), 3 8(6), and 3 8(22), respectively
deg	Denotes that the second solution is also degenerate Note that all degenerate solutions can be expressed in terms of the f notation, see 3 7(14)

Since the hypergeometric equation is symmetric in a and b, we assume that

(1) If a or b is an integer, then a is an integer
(2) If $c - a$ or $c - b$ is an integer, then $c - a$ is an integer
(3) If $b - a$ is an integer, then $b - a \geqslant 0$

If $a = c$, see 3 7(16)

Wronskians

Let

$$W_{ij} = W(w_i, w_j) = w_i(z)\, w_j'(z) - w_i'(z)\, w_j(z), \tag{27}$$

$$A = z^{-c}(1 - z)^{c-a-b-1} \tag{28}$$

Then

$$W_{12} = (1 - c)\, A \qquad W_{13} = -\frac{\Gamma(a + b + 1 - c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\, A, \tag{29}$$

$$W_{14} = -\frac{\Gamma(c + 1 - a - b)\,\Gamma(c)}{\Gamma(c - a)\,\Gamma(c - b)}\, A, \qquad W_{15} = \frac{\Gamma(1 + a - b)\,\Gamma(c)\, e^{i\pi c}}{\Gamma(a)\,\Gamma(c - b)}\, A, \tag{30}$$

$$W_{23} = -\frac{\Gamma(a+b+1-c)\,\Gamma(2-c)}{\Gamma(a+1-c)\,\Gamma(b+1-c)}A, \quad W_{24} = -\frac{\Gamma(c+1-a-b)\,\Gamma(2-c)}{\Gamma(1-a)\,\Gamma(1-b)}A, \tag{31}$$

$$W_{25} = -\frac{\Gamma(1+a-b)\,\Gamma(2-c)}{\Gamma(1-b)\,\Gamma(1+a-c)}A, \qquad W_{34} = -(c-a-b)\,A, \tag{32}$$

$$\begin{aligned} W_{35} &= \frac{\Gamma(a+b+1-c)\,\Gamma(1+a-b)\,e^{i\pi(c-b)}}{\Gamma(a+1-c)\,\Gamma(a)}A, \\ W_{45} &= \frac{\Gamma(c+1-a-b)\,\Gamma(1+a-b)\,e^{i\pi a}}{\Gamma(1-b)\,\Gamma(c-b)}A, \end{aligned} \tag{33}$$

$$W_{56} = (b-a)\,e^{i\pi c}A. \tag{34}$$

To get W_{i6} for $i = 1, 2, 3, 4$ simply interchange the roles of a and b in W_{i5}.

3.11. Kummer-Type Relations for the Logarithmic Solutions

3.11.1. Introduction

The Kummer relations for ${}_2F_1(a, b; c; z)$ are given by 3.8(1–4). Here we develop similar-type relations for the logarithmic solutions $V(z)$ and $W(z)$ [see 3.10(13, 20)]. The analysis is due to Nörlund (1963), and in the main it is convenient to adopt his notation. The connection between the notations is given by 3.11.2(7–11).

3.11.2. The Case Where c Is a Positive Integer

We assume that

$$c = s + 1, \qquad s \text{ a positive integer or zero,}$$

$$\text{neither } a \text{ nor } b \text{ is one of the numbers } 1, 2, \ldots, s. \tag{1}$$

Let

$$\begin{aligned} G_1(a, b; c; z) &= (-)^{s+1} z^{-s} (a)_{-s} (b)_{-s} s! \sum_{k=0}^{s-1} \frac{(a-s)_k (b-s)_k (s-1-k)!\,(-)^k z^k}{k!} \\ &= \frac{s}{z(a-1)(b-1)}\,{}_3F_2\left(\begin{matrix} 1-s, 1, 1 \\ 2-a, 2-b \end{matrix} \middle| z^{-1}\right), \end{aligned} \tag{2}$$

$$G(a, b, c, z) = G_1(a, b, c, z) + \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(s+1)_k k!}$$
$$\times \{\ln z + \psi(a+k) - \psi(a) + \psi(b+k) - \psi(b)$$
$$+ \psi(1+s) - \psi(1+s+k) - \psi(1+k) + \psi(1)\},$$
$$|z| < 1, \qquad |\arg z| < \pi, \tag{3}$$

$$g(a, b, c, z) = G_1(a, b, c, z) + \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(s+1)_k k!}$$
$$\times \{\ln z + \psi(a+k) + \psi(b+k) - \psi(1+s+k) - \psi(1+k)\},$$
$$|z| < 1, \qquad |\arg z| < \pi, \tag{4}$$

$$g_0(a, b, c, z) = G_1(a, b, c, z) + \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(s+1)_k k!}$$
$$\times \{\ln z + \psi(1-a-k) + \psi(1-b-k)$$
$$- \psi(1+s+k) - \psi(1+k)\},$$
$$|z| < 1, \qquad |\arg z| < \pi, \qquad |\arg(1-z)| < \pi, \tag{5}$$

$$g_1(a, b, c, z) = G_1(a, b, c, z) + \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(s+1)_k k!}$$
$$\times \{\ln(ze^{-i\epsilon\pi}) + \psi(1-a-k) + \psi(b+k)$$
$$- \psi(1+s+k) - \psi(1+k)\},$$
$$|z| < 1, \qquad \epsilon = \pm 1, \qquad -(1-\epsilon)\pi < \arg z < (1+\epsilon)\pi \tag{6}$$

We now identify the Norlund notation with our $V(z)$ and $W(z)$ notation In 3 10(1, 2) let $a_1 = b, a_2 = a, b_0 = 1, b_1 = c = s + 1$, i e, $V(z) = w_6$ For this set of parameters, instead of $V(z)$ and $V_1(z)$, write $V(a, b, c, z)$ and $V_1(a, b, c, z)$, respectively Then

$$V_1(a, b, c, z) = \frac{(-)^{s+1}\Gamma(1+b-a)}{s!\,\Gamma(1-a)\,\Gamma(b-s)} G_1(a, b, c, z), \tag{7}$$

$$V(a\ b, c, z) = (ze^{-i\pi})^{-b}\, {}_2F_1\left(\begin{matrix} b, 1+b-c \\ 1+b-a \end{matrix} \middle| \frac{1}{z}\right)$$
$$= \frac{(-)^{s+1}\Gamma(1+b-a)}{s!\,\Gamma(1-a)\,\Gamma(b-s)} g_1(a, b, c, z) \tag{8}$$

Again, in 3 10(3, 4), let $a_1 = a, a_2 = c - b, b_0 = 1, b_1 = c = s + 1$, and replace z by $1 - z$, that is, $W(z) = w_3$ For this set of parameters,

instead of $W(z)$ and $W_1(z)$, write $W(a, b; c; z)$ and $W_1(a, b; c; z)$, respectively. Then

$$W_1(a, b; c; z) = \frac{(-)^{s+1}\Gamma(a+b-s)}{s!\,\Gamma(a-s)\,\Gamma(b-s)}\, G_1(a, b; c; z), \tag{9}$$

$$V_1(a, b; c; z) = \frac{(-)^{s}\Gamma(1+a-b)\,\Gamma(a)}{\Gamma(1+s-a)\,\Gamma(a+b-s)}\, W_1(a, b; c; z), \tag{10}$$

$$\begin{aligned} W(a, b; c; z) &= {}_2F_1(a, b; 1+a+b-c; 1-z) \\ &= \frac{(-)^{s+1}\Gamma(a+b-s)}{s!\,\Gamma(a-s)\,\Gamma(b-s)}\, g(a, b; c; z). \end{aligned} \tag{11}$$

Note that each of the equations (3)–(6) satisfies 3.7(1). If either a or b is a negative integer or zero (4) is not defined. So that a and b can assume all possible values, except for those noted in (1), is the reason for considering the solutions g_0 and g_1. In (5) we suppose that neither a nor b is a positive integer, while in (6) we assume that a is not a positive integer and b is not an integer $\leqslant s$.

In view of 2.11(8),

$$\psi(a+k) + \psi(b+k) - \psi(1+s+k) - \psi(1+k) = (a+b-s-2)/k + O(k^{-2}),$$

so that (4) converges for $z = 1$ if $R(a+b) < s+2$ or if $a+b = s+2$.

It is easily shown that

$$\begin{aligned} g(a, b; c; z) &= G(a, b; c; z) + [\psi(a) + \psi(b) - \psi(c) - \psi(1)]\,{}_2F_1(a, b; c; z), \\ &\qquad |\arg z| < \pi, \end{aligned} \tag{12}$$

$$\begin{aligned} g_0(a, b; c; z) &= G(a, b; c; z) + [\psi(1-a) + \psi(1-b) - \psi(c) - \psi(1)]\,{}_2F_1(a, b; c; z), \\ &\qquad |\arg z| < \pi, \qquad |\arg(1-z)| < \pi, \end{aligned} \tag{13}$$

$$\begin{aligned} g_1(a, b; c; z) &= G(a, b; c; z) + [\psi(1-a) + \psi(b) - \psi(c) - \psi(1) - \epsilon i\pi]\,{}_2F_1(a, b; c; z), \\ &\qquad \epsilon = \pm 1, \qquad -(1-\epsilon)\pi < \arg z < (1+\epsilon)\pi, \end{aligned} \tag{14}$$

$$\begin{aligned} g_0(a, b; c; z) - g(a, b; c; z) &= \frac{\pi \sin \pi(a+b)}{\sin \pi a \sin \pi b}\,{}_2F_1(a, b; c; z), \\ &\qquad |\arg z| < \pi, \qquad |\arg(1-z)| < \pi, \end{aligned} \tag{15}$$

$$\begin{aligned} g_1(a, b; c; z) - g(a, b; c; z) &= \frac{\pi e^{-\delta i \pi a}}{\sin \pi a}\,{}_2F_1(a, b; c; z), \\ &\qquad \delta = \pm 1, \qquad -(1-\delta)\pi/2 < \arg z < (1+\delta)\pi/2, \end{aligned} \tag{16}$$

$$g_0(a, b, c, z) - g_1(a, b, c, z) = \frac{\pi e^{\delta i \pi b}}{\sin \pi b} {}_2F_1(a, b, c, z)$$

$$\delta = \pm 1, \qquad -(1 - \delta)\pi/2 < \arg z < (1 + \delta)\pi/2 \tag{17}$$

By differentiation of the power series, we have

$$\frac{d^n}{dz^n} g(a, b, c, z) = \frac{(a)_n (b)_n}{(c)_n} g(a + n, b + n, c + n, z), \tag{18}$$

$$\frac{d^n}{dz^n} \{z^{a+n} {}^1g(a, b, c, z)\} = (a)_n z^a {}^1g(a + n, b, c, z), \tag{19}$$

$$\frac{d^n}{dz^n} \{z^c {}^1g(a, b, c, z)\} = (-)^n (1 - c)_n z^{c-n} {}^1g(a, b, c - n, z) \tag{20}$$

Equations (18)–(20) are the analogs of 3.4(1, 4, 5), respectively, suitably specialized. Clearly in (18)–(20), g may be replaced by g_0, g_1, or G.

It is readily verified that both $G(a, b, c, z)$ and $(1 - z)^{-b} G(c - a, b, c, z/(z - 1))$ satisfy the differential equation 3.7(1). Further, for z small, the coefficient of z^{-a} is the same for both solutions. Hence, there must be a relation of the form

$$(1 - z)^{-b} G(c - a, b, c, z/(z - 1)) - G(a, b, c, z) = C\, {}_2F_1(a, b, c, z), \tag{21}$$

where C is a constant. Expand both sides of (21) in powers of z and equate the constant terms. We find that

$$C = -\delta i \pi + A_s,$$

where δ is as in (16), and A_s can be put in the form

$$A_s = \frac{s}{a - 1} {}_3F_2\left(\begin{matrix} 1 - s, 1, 1 \\ 2 - a, 2 \end{matrix} \middle| 1\right) = \psi(1 - a) - \psi(1 - a + s)$$

in view of 3.13.3(42). Thus,

$$G(a, b, c, z) = (1 - z)^{-b} G(c - a, b, c, z/(z - 1))$$
$$+ [\delta i \pi + \psi(1 - a + s) - \psi(1 - a)]\, {}_2F_1(a, b, c, z),$$
$$\delta = \pm 1, \qquad -(1 - \delta)\pi/2 < \arg z < (1 + \delta)\pi/2 \tag{22}$$

Now $G(a, b, c, z)$ is symmetric in the parameters a and b. So

$$G(a, b, c, z) = (1 - z)^{-a} G(a, c - b, c, z/(z - 1))$$
$$+ [\delta i \pi + \psi(1 - b + s) - \psi(1 - b)]\, {}_2F_1(a, b, c, z) \tag{23}$$

and combining the latter two formulas, we get

$$G(a, b; c; z) = (1 - z)^{c-a-b}G(c - a, c - b; c; z) + [\psi(1 - a + s) - \psi(1 - a) + \psi(1 - b + s) - \psi(1 - b)]\, {}_2F_1(a, b; c; z), \qquad |\arg z| < \pi, \tag{24}$$

and in particular if $c = 1$, i.e., $s = 0$, we have

$$G(a, b; 1; z) = (1 - z)^{1-a-b}G(1 - a, 1 - b; 1; z), \qquad |\arg z| < \pi. \tag{25}$$

The combination (12) and (22) produces

$$g(a, b; c; z) = (1 - z)^{-b}g\left(c - a, b; c; \frac{z}{z - 1}\right) - \frac{\pi e^{-\delta i\pi a}}{\sin \pi a}\, {}_2F_1(a, b; c; z), \tag{26}$$

a not an integer and δ as in (22). If we use (16), (26) becomes

$$g_1(a, b; c; z) = (1 - z)^{-b}g(c - a, b; c; z/(z - 1)). \tag{27}$$

In a similar fashion, we see that

$$g_1(b, a; c; z) = (1 - z)^{-a}g(a, c - b; c; z/(z - 1)), \tag{28}$$

$$g_1(a, b; c; z) = (1 - z)^{-a}g_0(a, c - b; c; z/(z - 1)), \tag{29}$$

$$g_1(b, a; c; z) = (1 - z)^{-b}g_0(c - a, b; c; z/(z - 1)), \tag{30}$$

$$g(a, b; c; z) = (1 - z)^{-b}g_1(c - a, b; c; z/(z - 1)), \tag{31}$$

$$g_0(a, b; c; z) = (1 - z)^{-a}g_1(a, c - b; c; z/(z - 1)). \tag{32}$$

Combining these formulas, we get

$$g(a, b; c; z) = (1 - z)^{c-a-b}g_0(c - a, c - b; c; z), \tag{33}$$

$$g_0(a, b; c; z) = (1 - z)^{c-a-b}g(c - a, c - b; c; z), \tag{34}$$

$$g_1(a, b; c; z) = (1 - z)^{c-a-b}g_1(c - b, c - a; c; z). \tag{35}$$

It can be shown that $g(a, b; c; z)$ and ${}_2F_1(a, b; c; z)$ satisfy the same contiguous relations 3.4(19–27). For this and related items, see Kovalenko (1967).

Finally, we record Mellin–Barnes integral representations for the functions g, g_0, and g_1:

$$g(a, b; c; z) = \frac{(-)^{s+1}s!}{2\pi i\Gamma(a)\,\Gamma(b)} \int_{-i\infty}^{i\infty} \Gamma(a - t)\,\Gamma(b - t)\,\Gamma(t)\,\Gamma(t - s)\, z^{-t}\, dt, \qquad |\arg z| < 2\pi, \tag{36}$$

where the path of integration is indented if necessary so that the poles of $\Gamma(t)\Gamma(t - s)$ lie to the left and the poles of $\Gamma(a - t)\Gamma(b - t)$ lie to

the right of it Computing the integral as $2\pi i$ times the sum of the residues of the integrand due to the simple poles at $t = s - 1, s - 2, \ldots, 1$, and due to the double poles at $t = 0, -1, -2, \ldots$, we get the series (4)

The representation (11) is equivalent to

$$g(a, b, c, z) = \frac{(-)^{s+1} s!\, \Gamma(a - s)\, \Gamma(b - s)}{2\pi i \Gamma(a)\, \Gamma(b)} \times \int_{-i\infty}^{i\infty} \frac{\Gamma(a - t)\, \Gamma(b - t)\, \Gamma(t)(z - 1)^{-t}}{\Gamma(a + b - s - t)}\, dt,$$
$$|\arg(z - 1)| < \pi, \qquad (37)$$

where the contour passes between the poles of $\Gamma(t)$ and the poles of $\Gamma(a - t)\Gamma(b - t)$ This follows by evaluating the integral as $2\pi i$ times the sum of the residues of the integrand due to the poles at $t = 0, -1, -2, \ldots$

We also have

$$g(a, b, c, z) = \frac{(-)^{s+1} s!\, z^{-s}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a - t)\, \Gamma(b - t)\, \Gamma(t - s)(z - 1)^{s-t}}{\Gamma(a + b - t)}\, dt,$$
$$|\arg(z - 1)| < \pi, \qquad (38)$$

where the path of integration separates the poles of $\Gamma(a - t)$ and $\Gamma(b - t)$ from those of $\Gamma(t - s)$ This is again readily proved using the residue calculus and (4)

Next we have

$$g_0(a, b, c, z) = \frac{(-)^{s+1} s!\, \Gamma(1 - a)\, \Gamma(1 - b)}{2\pi i} \times \int_{s+1-i\infty}^{s+1+i\infty} \frac{\Gamma(t)\, \Gamma(t - s)\, z^{-t}}{\Gamma(1 + t - a)\, \Gamma(1 + t - b)}\, dt,$$
$$0 < z < 1, \qquad R(a + b) < s + 2 \qquad (39)$$

Here the path of integration is a straight line parallel to the imaginary axis and crossing the real axis at $s + 1$ Evaluate the integral in (39) as the sum of residues of its integrand at the poles, $t = s, s - 1, \ldots$, to get (5) Note that the integral in (39) vanishes for $z > 1$, and diverges if z is negative or complex

In a similar fashion,

$$g_1(a, b, c, z) = \frac{(-)^{s+1} s!\, \Gamma(1 - a)}{2\pi i \Gamma(b)} \times \int_{s+1-i\infty}^{s+1+i\infty} \frac{\Gamma(t)\, \Gamma(t - s)\, \Gamma(b - t)(-z)^{-t}}{\Gamma(t - a + 1)}\, dt,$$
$$|\arg(-z)| < \pi \qquad (40)$$

In this instance the contour is a line parallel to the imaginary axis so that the poles at $t = b, b + 1,\ldots,$ lie to the right of the path while those at $t = s, s - 1,\ldots,$ lie to the left of the path. Application of the Cauchy residue calculus leads to the representations (6) and (8).

3.11.3. The Case Where c Is a Negative Integer or Zero

Recall that if c is not an integer, w_1 and w_2 are linearly independent solutions of 3.7(1). If c is a negative integer or zero, and both a and b never coincide with the numbers $0, -1,\ldots, c$, then the Kummer-type relations follow from those of the previous section with an appropriate change of notation as the roles of w_1 and w_2 are interchanged.

Let us now suppose that c is a negative integer, and that either a or b is one of the numbers $0, -1,\ldots, c$. In particular let

$$c = 1 - s, \qquad a = -m, \qquad m < s,$$

$$s \text{ a positive integer}, \qquad m \text{ a positive integer or zero}. \tag{1}$$

If b is also one of the numbers $0, -1,\ldots, c$, then assume $-b \leqslant m$. Let

$$f(a, b; c; z) = \sum_{k=0}^{-a} \frac{(a)_k (b)_k z^k}{(c)_k k!} = \sum_{k=0}^{m} \frac{(-m)_k (b)_k z^k}{(1 - s)_k k!}. \tag{2}$$

Also, see 3.7(14),

$$f(a, b; c; z) = \frac{(s - 1 - m)!\,(b)_m z^m}{(s - 1)!}\, {}_2F_1\left(\begin{matrix} -m, s - m \\ 1 - m - b \end{matrix} \middle| z^{-1}\right),$$

$$b \neq 1, \qquad 1 - m - b \quad \text{is not a negative integer or zero}. \tag{3}$$

Now both (2) and $(1 - z)^{-a} f(a, c - b; c; z/(z - 1))$ satisfy 3.7(1). As w_2 is also a solution, there exists a relation of the form

$$\begin{aligned}(1 - z)^{-a} f(a, c - b; c; z/(z - 1)) &- f(a, b; c; z) \\ &= C z^{1-c}\, {}_2F_1(1 + a - c, 1 + b - c; 2 - c; z),\end{aligned} \tag{4}$$

where C is a constant. If we expand and equate like powers of z^{1-c}, we find

$$C = \frac{(b)_s}{s!} \sum_{k=0}^{s-m-1} \frac{(m + 1 - s)_k (-s)_k}{(1 - s)_k k!} = \frac{(-)^{s-m-1} (b)_s (s - m - 1)!\, m!}{s!\,(s - 1)!} \tag{5}$$

in view of 3.13.1(5). Similarly,

$$(1 - z)^{-a} f(a, c - b; c; z/(z - 1)) = f(a, b; c; z), \quad a = 0, -1,\ldots, c, \ b \text{ arbitrary}. \tag{6}$$

Thus from (4) and (6),

$$(1-z)^{c-a-b}f(c-a, c-b, c, z) - f(a, b, c, z)$$
$$= Cz^{1-c}{}_2F_1(1+a-c, 1+b-c, 2-c, z), \tag{7}$$

C as in (5), and therefore

$$f(a, b, c, z) = (1-z)^{-b}f(b, c-a, c, z/(z-1))$$
$$= (1-z)^{c-a-b}f(c-a, c-b, c, z) \tag{8}$$

where both a and b are one (but not necessarily the same) of the numbers $0, -1, \ldots, c$

3.12. Quadratic Transformations

The formulas 3.8(1–24) may be viewed as a transformation of the ${}_2F_1$ in the variable, actually a linear fractional transformation. Put another way, Kummer's solutions arise from a study of transformations of 3.7(1) into itself under projective transformations of the independent variable. It is natural to inquire if there exist quadratic and higher transformations. It can be shown that if and only if the numbers

$$\pm(1-c), \qquad \pm(a-b), \qquad \pm(a+b-c)$$

are such that one of them is $\frac{1}{2}$ or that two of them are equal, then a quadratic transformation exists. The basic formulas are due to Gauss and Kummer and a complete list is due to Goursat, see Erdélyi *et al* (1953, Vol. 1, pp. 64–68, 110–113). Some examples follow.

$${}_2F_1(2a, 2a-c+1, c, z) = (1+z)^{-2a}{}_2F_1\left(a, a+\tfrac{1}{2}, c, \frac{4z}{(1+z)^2}\right) \tag{1}$$

$${}_2F_1(2a, 2a-c+1, c, z) = (1-z)^{-2a}{}_2F_1\left(a, c-a-\tfrac{1}{2}, c, -\frac{4z}{(1-z)^2}\right), \tag{2}$$

$${}_2F_1(a, a+\tfrac{1}{2}(1-c), \tfrac{1}{2}(c+1), z^2) = (1+z)^{-2a}{}_2F_1\left(a, \tfrac{1}{2}c, c, \frac{4z}{(1+z)^2}\right), \tag{3}$$

$${}_2F_1(2a, 2b, a+b+\tfrac{1}{2}, z) = {}_2F_1(a, b, a+b+\tfrac{1}{2}, 4z(1-z)), \tag{4}$$

$${}_2F_1(2a-1, 2b-1, a+b-\tfrac{1}{2}, z) = (1-2z){}_2F_1(a, b, a+b-\tfrac{1}{2}, 4z(1-z)) \tag{5}$$

$${}_2F_1(a, 1-a, c, z) = (1-z)^{c-1}{}_2F_1(\tfrac{1}{2}(c-a), \tfrac{1}{2}(c+a-1), c, 4z(1-z)) \tag{6}$$

$$ {}_2F_1(a, b; 2a; z) = \left(\frac{2}{1+y}\right)^{2b} {}_2F_1\left(b, b+\tfrac{1}{2}-a; a+\tfrac{1}{2}; \left(\frac{1-y}{1+y}\right)^2\right), $$
$$ y = (1-z)^{1/2}, \quad (7) $$

$$ {}_2F_1(a, b; 2a; 2z) = (1-z)^{-b}\, {}_2F_1\left(\tfrac{1}{2}b, \tfrac{1}{2}(b+1); a+\tfrac{1}{2}; \left(\frac{z}{1-z}\right)^2\right). \quad (8) $$

Sometimes (4)–(6) are given with $2z$ and $4z(1-z)$ replaced by $(1-y)$ and z, respectively, y as in (7). All formulas are valid near $z = 0$ and $(1-z)^\delta > 0$ if δ is real and $z < 1$. Many other forms can be derived from (1)–(8) by use of Kummer's formulas 3.8(1–4).

For the proof of (1), recall that ${}_2F_1(a, b; c; x)$ satisfies

$$ x(1-x)\frac{d^2v}{dx^2} + [c-(a+b+1)x]\frac{dv}{dx} - abv = 0. \quad (9) $$

If we write

$$ x = \frac{4z}{(1+z)^2}, \qquad v = (1+z)^{2a}w, $$

Eq. (9) becomes

$$ z(1-z)\frac{d^2w}{dz^2} + [c-(4b-2c)z+(c-4a-2)z^2]\frac{dw}{dz} $$
$$ - 2a[2b-c+(2a-c+1)z]w = 0. \quad (10) $$

With $b = a + \frac{1}{2}$, the latter reduces to

$$ z(1-z)\frac{d^2w}{dz^2} + [c-(4a-c+2)z]\frac{dw}{dz} - 2a(2a+1-c)w = 0, \quad (11) $$

which is satisfied by ${}_2F_1(2a, 2a+1-c; c; z)$. It follows that both sides of (1) satisfy the same differential equation and that at $z = 0$, both sides of (1) have the same value. If c is not a negative integer or zero, the differential equations (9) and (11) have only one solution which is regular at $z = 0$ and (1) follows. Now apply 3.8(3) to the right-hand side of (1) to establish (2). If in (10) we put $b = c/2$ and replace z^2 by x, we obtain a differential equation satisfied by the left-hand side of (3). Then, provided that c is not a negative integer or zero, a similar argument for the proof of (1) leads to the statement (3). The differential equation approach can also be used to prove (4)–(8). Equation (4) may also be derived from (2). There replace c by $a + b + \frac{1}{2}$, z by $z/(z-1)$, and apply 3.8(2) to the left-hand side. A generalization of (7) is given by 9.5(1).

The combination (4) and 3 9(7) yields

$$
{}_2F_1\left(\begin{matrix}2a, 2b\\ a+b+\frac{1}{2}\end{matrix}\middle|\frac{z+1}{2}\right) = \frac{\Gamma(a+b+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\,\Gamma(b+\frac{1}{2})}\,{}_2F_1\left(\begin{matrix}a, b\\ \frac{1}{2}\end{matrix}\middle|z^2\right)
$$

$$
+ 2z\,\frac{\Gamma(a+b+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(a)\,\Gamma(b)}\,{}_2F_1\left(\begin{matrix}a+\frac{1}{2}, b+\frac{1}{2}\\ \frac{3}{2}\end{matrix}\middle|z^2\right) \tag{12}
$$

For cubic and higher order transformations, see Erdélyi *et al* (1953, Vol I, p 67), and the references given there

Quadratic transformations for the logarithmic solutions of (9) and (11) have been discussed by Nörlund (1963), see 3 11 In the sequel $c = s + 1$, s a positive integer or zero, whenever the functions G, g, g_0, or g_1 are involved, and $c = 1 - s$, s a positive integer, whenever the function f is involved Now $G(a, a + \frac{1}{2}, c, x)$ satisfies (9) with $b = a + \frac{1}{2}$ and $G(2a, 2a + 1 - c, c, z)$ satisfies (11) Hence, if neither a nor b is one of the numbers $1, 2, \ldots, s$,

$$
G\left(a, a+\tfrac{1}{2}, c, \frac{4z}{(1+z)^2}\right) = (1+z)^{2a}[C_1 G(2a, 2a-c+1, c, z)
$$

$$
+ C_2 F_1(2a, 2a-c+1, c, z)] \tag{13}
$$

where C and C_1 are constants Divide both sides of (13) by z^{1-c} or by $\ln z$ if $c = 1$ Let $z \to 0$ and find $C_1 = 1$ Now let $z \to 1$, and use 3 13 1(1, 9) Then

$$
C = \psi(1+s-2a) - \psi(1-2a) + 2\ln 2, \qquad R(2a) < s + \tfrac{1}{2} \tag{14}
$$

In a similar fashion, if $R(2a) > s + \frac{1}{2}$, then use of 3 13 1(2, 12) gives C as in (14) Hence,

$$
G\left(a, a+\tfrac{1}{2}, c, \frac{4z}{(1+z)^2}\right) = (1+z)^{2a}[G(2a, 2a-c+1, c, z)
$$

$$
+ C\,{}_2F_1(2a, 2a-c+1, c, z)]
$$

$$
2a \neq 1, 2, \ldots, 2s, \tag{15}
$$

and with the aid of 3 11 2(12) or 3 11 2(13), we have

$$
g\left(a, a+\tfrac{1}{2}, c, \frac{4z}{(1+z)^2}\right) = (1+z)^{2a} g(2a, 2a-c+1, c, z)
$$

$$
= (1+z^{-1})^{2a} g(2a, 2a-c+1, c, z^{-1}),
$$

$$
2a \neq 1, 2, \ldots, 2s \tag{16}
$$

and a like relation with g replaced by g_0 under the stronger restriction that $2a$ is not a positive integer. Note that $(4z)/(1+z)^2$ is unchanged if z is replaced by $1/z$. From 3.11.2(16), (1), and (16), we have

$$(1+z)^{-2a} g_1\left(a, a+\tfrac{1}{2}; c; \frac{4z}{(1+z)^2}\right) = g_1(2a, 2a-c+1; c; z) + \frac{\pi}{\sin 2\pi a}\, {}_2F_1(2a, 2a-c+1; c; z), \quad 2a \text{ not an integer.} \tag{17}$$

Next we consider the rational solutions, and so suppose that c is a nonpositive integer, that is, $c = 1 - s$, s a positive integer. Now (9) has the solution $f(a, a+\frac{1}{2}; c; x)$, and (11) has the solutions $f(2a, 2a-c+1; c; z)$ and $z^{1-c}\, {}_2F_1(2a-c+1, 2a-2c+2; 2-c; z)$, provided that $2a$ is one of the numbers $0, -1, \ldots, -(2s-1)$. Hence, we have a relation of the form

$$\begin{aligned}(1+z)^{-2a} f\left(a, a+\tfrac{1}{2}; c; \frac{4z}{(1+z)^2}\right) &= C_1 f(2a, 2a-c+1; c; z) \\ &\quad + C_2 z^{1-c}\, {}_2F_1(2a-c+1, 2a-2c+2; 2-c; z).\end{aligned} \tag{18}$$

Let $z \to 0$, then the coefficient of the ${}_2F_1$ vanishes and each of the f functions approach unity, and so $C_1 = 1$. To determine C_2, we consider two cases. First suppose that $2a$ is one of the numbers $0, -1, \ldots, 1-s$. Then the left side and the coefficient of C_1 on the right side of (18) are polynomials in z of degree $-2a$, but the coefficient of C_2 contains higher powers of z. Hence, $C_2 = 0$, and

$$\begin{aligned}(1+z)^{-2a} f\left(a, a+\tfrac{1}{2}; c; \frac{4z}{(1+z)^2}\right) &= f(2a, 2a-c+1; c; z) \\ &= z^{-2a} f(2a, 2a-c+1; c; z^{-1}),\end{aligned}$$
$$c = 1-s, \qquad s = 1, 2, \ldots, \qquad 2a = 0, -1, \ldots, 1-s. \tag{19}$$

Now assume that $2a$ is one of the numbers $-s, -s-1, \ldots, -(2s-1)$. Then the ${}_2F_1$ in (18) is a polynomial in z of degree $(c+1-2a)$, and it can be seen that

$$\begin{aligned}&z^{1-c}\, {}_2F_1(2a-c+1, 2a-2c+2; 2-c; z) \\ &\quad = \frac{(-)^{c-2a-1}\Gamma(1-c)\,\Gamma(2-c)}{\Gamma(2a-2c+2)\,\Gamma(1-2a)}\, z^{-2a} f(2a, 2a-c+1; c; z^{-1}).\end{aligned} \tag{20}$$

Multiply both sides of (18) by z^{2a} and let $z \to \infty$ Then C_2 is the reciprocal of the coefficient of $z^{-2a}f$ on the right side of (20) Hence,

$$(1+z)^{-2a}f\left(a\ a+\tfrac{1}{2}, c, \frac{4z}{(1+z)^2}\right)$$
$$= f(2a\ 2a-c+1\ c, z) + (-)^{c-2a}\ {}_1\frac{\Gamma(2a-2c+2)\,\Gamma(1-2a)}{\Gamma(1-c)\,\Gamma(2-c)}$$
$$\times z^{1-c}\,{}_2F_1(2a-c+1, 2a-2c+2, 2-c, z)$$

$c = 1-s$, $s = 1, 2, \ \ , 2a$ is a negative integer, $s-1 < -2a \leqslant 2s-1$, (21)

and under the same conditions,

$$f\left(a\ a+\tfrac{1}{2}, c, \frac{4z}{(1+z)^2}\right) = (1+z)^{2a}f(2a\ 2a-c+1, c, z)$$
$$+ (1+1/z)^{2a}f(2a, 2a-c+1, c, 1/z) \qquad (22)$$

Consider now the transformations for the logarithmic solutions suggested by (2) Thus from 3 11 2(23), we have

$$G\left(a, a+\tfrac{1}{2}, c, \frac{4z}{(1+z)^2}\right) = \left(\frac{1+z}{1-z}\right)^{2a} G\left(a, c-a-\tfrac{1}{2}, c, -\frac{4z}{(1-z)^2}\right)$$
$$+ [\delta i\pi + \psi(\tfrac{1}{2}-a+s) - \psi(\tfrac{1}{2}-a)]$$
$$\times\ {}_2F_1\left(a\ a+\tfrac{1}{2}, c, \frac{4z}{(1+z)^2}\right),$$
$$\delta = \pm 1, \qquad -(1-\delta)\pi/2 < \arg z < (1+\delta)\pi/2, \qquad (23)$$

where $c = 1+s$, s a positive integer or zero The combination (15), (23) gives

$$G\left(a, c-a-\tfrac{1}{2}, c, -\frac{4z}{(1-z)^2}\right) = (1-z)^{2a}G(2a, 2a-c+1, c, z)$$
$$+ [-\delta i\pi + 2\ln 2 + \psi(s+1-2a)$$
$$- \psi(1-2a)$$
$$- \psi(s+\tfrac{1}{2}-a) + \psi(\tfrac{1}{2}-a)]$$
$$\times\ {}_2F_1\left(a, c-a-\tfrac{1}{2}, c, -\frac{4z}{(1+z)^2}\right), \qquad (24)$$

provided that $2a \neq 1, 2, \ \ , 2s$ Application of the transformation 3 11 2(31) to (16) yields

$$g_1\left(c-a-\tfrac{1}{2}\ a, c, -\frac{4z}{(1-z)^2}\right) = (1-z)^{2a}g(2a, 2a-c+1, c, z) \qquad (25)$$

provided that $2a \neq 1, 2, \ldots, 2s$. If $2a$ is not a positive integer, we can replace g by g_0 in (25). Similarly 3.11.2(32) and (17) give

$$g_0\left(a, c-a-\tfrac{1}{2}; c; -\frac{4z}{(1-z)^2}\right) = (1-z)^{2a} g_1(2a, 2a-c+1; c; z) - (1-1/z)^{2a} g_1(2a, 2a-c+1; c; 1/z), \quad 2a \text{ not an integer}. \tag{26}$$

Let $c = 1 - s$ and use 3.11.3(4). Then from (19) and (22), we get

$$f\left(a, c-a-\tfrac{1}{2}; c; -\frac{4z}{(1-z)^2}\right) = (1-z)^{2a} f(2a, 2a-c+1; c; z) + (1-1/z)^{2a} f(2a, 2a-c+1; c; 1/z), \quad -2a = s, s+1, \ldots, 2s-1, \tag{27}$$

$$f\left(a, c-a-\tfrac{1}{2}; c; -\frac{4z}{(1-z)^2}\right) = (1-z)^{2a} f(2a, 2a-c+1; c; z) = (1-1/z)^{2a} f(2a, 2a-c+1; c; 1/z), \quad 2a \text{ is a nonpositive integer} \geqslant 1-s, \tag{28}$$

$$f\left(a, c-a-\tfrac{1}{2}; c; -\frac{4z}{(1-z)^2}\right) = (1-z)^{2c-2a-1} f(c-2a, 2c-2a-1; c; z) + (1-1/z)^{2c-2a-1} f(c-2a, 2c-2a-1; c; 1/z), \quad 2a \text{ is an odd integer}, \quad 0 > 2a \geqslant 1-s. \tag{29}$$

Finally we consider relations analogous to (3) and (4). We state results and omit details as the proofs are much akin to those above.

$$g\left(a, \tfrac{1}{2}c; c; \frac{4z}{(1+z)^2}\right) = \tfrac{1}{2}(1+z)^{2a} g(a, a+\tfrac{1}{2}(1-c); \tfrac{1}{2}(1+c); z^2), \tag{30}$$

where c is an odd positive integer, a is not an integer $< c$.

$$f\left(a, \tfrac{1}{2}c; c; \frac{4z}{(1+z)^2}\right) = (1+z)^{2a} f(a, a+\tfrac{1}{2}(1-c); \tfrac{1}{2}(1+c); z^2) + (1+1/z)^{2a} f(a, a+\tfrac{1}{2}(1-c); \tfrac{1}{2}(1+c); 1/z^2), \tag{31}$$

where c is an odd negative integer, $a = \tfrac{1}{2}(c-1), \tfrac{1}{2}(c-3), \ldots, c$.

$$f\left(a, \tfrac{1}{2}c; c; \frac{4z}{(1+z)^2}\right) = (1+z)^{2a} f(a, a+\tfrac{1}{2}(1-c); \tfrac{1}{2}(1+c); z^2) = (1+1/z)^{2a} f(a, a+\tfrac{1}{2}(1-c); \tfrac{1}{2}(1+c); 1/z^2), \tag{32}$$

where c is an odd negative integer, $a = 0, -1, \ldots, \frac{1}{2}(1 + c)$

$$g(2a, 2b, a + b + \tfrac{1}{2}, \tfrac{1}{2}[1 - (1 - z)^{1/2}]) = g(a, b, a + b + \tfrac{1}{2}, z) + (\pi/\sin 2\pi a)\, {}_2F_1(a, b, a + b + \tfrac{1}{2}, z), \tag{33}$$

where $c = a + b + \frac{1}{2}$ is a positive integer, $2a$ is not an integer

$$G(2a, 2b, a + b + \tfrac{1}{2}, \tfrac{1}{2}[1 - (1 - z)^{1/2}]) = G(a, b, a + b + \tfrac{1}{2}, z) + v\, {}_2F_1(a, b, a + b + \tfrac{1}{2}; z),$$

where $c = a + b + \frac{1}{2}$ is a positive integer, $2a \neq 1, 2, \ldots, 2c - 2$,

$$v = \psi(a) + \psi(b) - \psi(2a) - \psi(2b) + (\pi/\sin 2\pi a) = \sum_{k=1}^{2c-2} \frac{(-)^k}{2a - k} - 2\ln 2 = \sum_{k=1}^{2c-2} \frac{(-)^k}{2b - k} - 2\ln 2. \tag{34}$$

$$g(2a, 2b, a + b + \tfrac{1}{2}, \tfrac{1}{2}[1 + (1 - z)^{1/2}]) = -(\pi/\sin 2\pi a)\, {}_2F_1(a, b, a + b + \tfrac{1}{2}, z), \tag{35}$$

where c and a are as in (33)

$$f(2a, 2b, c, \tfrac{1}{2}[1 - (1 - z)^{1/2}]) = f(a, b, c, z) + Cz^{1-c}\, {}_2F_1(1 + a - c, 1 + b - c, 2 - c; z),$$

where $c = a + b + \frac{1}{2}$ is a negative integer, $2a = 0, -1, \ldots, 2c - 1$, $C = 0$ if a or b is one of the numbers $0, -1, \ldots, -[|c|/2]$, and

$$C = -\frac{\Gamma(\frac{1}{2} - a)\,\Gamma(\frac{1}{2} - b)\,\Gamma(1 - a)\,\Gamma(1 - b)}{\pi\Gamma(1 - c)\,\Gamma(2 - c)} \quad \text{if } a = -\tfrac{1}{2}, -\tfrac{3}{2}, \ldots, \tfrac{1}{2} + [\tfrac{1}{2}c] \tag{36}$$

$$f(2a, 2b, c, \tfrac{1}{2}[1 - (1 - z)^{1/2}]) = (1 - z)^{1/2} f(a + \tfrac{1}{2}, b + \tfrac{1}{2}, c, z), \tag{37}$$

where c is as in (36), and a or b is one of the numbers $-\frac{1}{2}, -\frac{3}{2}, \ldots, \frac{1}{2} + [\frac{1}{2}c]$

$$f(2a, 2b, c; \tfrac{1}{2}[1 - (1 - z)^{1/2}]) = \frac{\Gamma(\frac{1}{2} - b)\,\Gamma(1 - b)}{\Gamma(1 - c)\,\Gamma(a - b + 1)} (z^{-1}e^{i\pi})^a \times {}_2F_1(a, \tfrac{1}{2} - b, a - b + 1, 1/z), \tag{38}$$

where c is as in (36), $2a = 0, -1, \ldots, c$, and $0 < \arg z < 2\pi$

3.13. The ${}_{p+1}F_p$ for Special Values of the Argument

3.13.1. The Regular and Logarithmic Solutions of the Gaussian Differential Equation Near $z = 1$

Put $z = 1$ in 3.6(1). The integral is readily evaluated using 2.6(3), and so

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, $$

$$ c \text{ not a negative integer or zero,} \qquad R(c-a-b) > 0. \tag{1} $$

From 3.8(2) and (1),

$$ \lim_{z\to 1} (1-z)^{a+b-c}\, {}_2F_1(a, b; c; z) = \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}, $$

$$ c \text{ not a negative integer or zero,} \qquad R(a+b-c) > 0. \tag{2} $$

We also have

$$ {}_2F_1(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}, \tag{3} $$

$$ {}_2F_1(-n, n+\lambda; c; 1) = \frac{(-)^n(1+\lambda-c)_n}{(c)_n}, \tag{4} $$

where n is a positive integer or zero and c is not a negative integer or zero. If c is a negative integer or zero, say $c = -m$, and $m > n$, then

$$ \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{(-m)_k k!} = \frac{(m-n)!\,(m+b+1-n)_n}{m!}, \tag{5} $$

which is known as Vandermonde's theorem.

We next consider the behavior of the logarithmic solutions to 3.7(1) as $z \to 1$. From 3.11.2(11),

$$ g(a, b; c; 1) = \frac{(-)^{s+1} s!\,\Gamma(a-s)\,\Gamma(b-s)}{\Gamma(a+b-s)} \tag{6} $$

and using 3.11.2(34), we have

$$ \lim_{z\to 1} (1-z)^{a+b-c} g_0(a, b; c; z) = \frac{(-)^{s+1} s!\,\Gamma(1-a)\,\Gamma(1-b)}{\Gamma(1+s-a-b)}. \tag{7} $$

Here and throughout the remainder of this section except in (16),

$c = s + 1$, s a positive integer or zero Combining 3 11 2(8) and (1), we obtain

$$g_1(a, b\ c, 1) = \frac{(-)^{s+1} s!\, \Gamma(b - s)\, \Gamma(1 + s - a - b)}{\Gamma(1 + s - a)} e^{\pm i\pi b}, \quad R(a + b) < 1 + s, \tag{8}$$

where the upper or lower sign is taken according as z tends to unity from the upper or lower edge of the cut from 0 to ∞ From 3 11 2(13),

$$G(a\ b, c, 1) = \frac{s!\, \Gamma(1 + s - a - b)}{\Gamma(1 + s - a)\, \Gamma(1 + s - b)} [\psi(1 + s) + \psi(1) - \psi(1 - a) - \psi(1 - b)] \tag{9}$$

where neither a nor b is a positive integer, $R(a + b) < 1 + s$ If a is a positive integer $> s$ and b is restricted as in (9), then the right-hand side of (9) is defined in view of 2 4(13) In particular,

$$G(c + n\ b, c, 1) = \frac{(-)^{s+1} s!\, n!\, \Gamma(b - s)}{\Gamma(1 + b + n)} \quad n = 0, 1, 2, \ \ , \ R(b) < -n \tag{10}$$

It is clear from (1) that ${}_2F_1(1 + n + s, b, 1 + s, z)$ vanishes when $z \to 1$ provided that $R(n + b) < 0$, and hence from 3 11 2(12),

$$g(c + n, b\ c, 1) = G(c + n, b, c; 1) \qquad R(n + b) < 0 \tag{11}$$

In a similar fashion, the combination (2), (9), and 3 11 2(24) yields

$$\lim_{z \to 1} (1 - z)^{a+b-c} G(a, b, c, z) = \frac{s!\, \Gamma(a + b - 1 - s)}{\Gamma(a)\, \Gamma(b)} \times [\psi(1 + s) + \psi(1) - \psi(a) - \psi(b)], \quad R(a + b) > 1 + s \tag{12}$$

Next we turn to the situation when $a + b = 1 + s$ If $s = 0$ in 3 11 2(11), then with the aid of 3 11 2(4), we find

$$\lim_{z \to 1} \frac{{}_3F_1(a\ b\ a + b, z)}{\ln(1 - z)} = -\frac{\Gamma(a + b)}{\Gamma(a)\, \Gamma(b)}, \tag{13}$$

and from 3 11 2(12),

$$\lim_{z \to 1} \frac{G(a\ b, a + b, z)}{\ln(1 - z)} = \frac{s!}{\Gamma(a)\, \Gamma(b)} [\psi(a) + \psi(b) - \psi(1 + s) - \psi(1)] \tag{14}$$

From 3.11.2(4)

$$\lim_{z\to 0} z^{c-1}g(a, b; c; z) = (-)^{s+1}s!\,(s-1)!\,(a)_{-s}(b)_{-s} \qquad \text{if} \quad s > 0,$$
$$\lim_{z\to 0} \frac{g(a, b; c; z)}{\ln z} = 1 \qquad \text{if} \quad s = 0. \tag{15}$$

Finally, using 3.11.3(2) and (5), we have

$$f(a, b; c; 1) = \frac{(s-1-m)!\,\Gamma(b+s)}{(s-1)!\,\Gamma(b+s-m)}, \tag{16}$$

where $c = 1 - s$, $a = -m$, $m + 1 < s$, and $(s - 1)$ and m are positive integers.

3.13.2. Special Values Related to the Quadratic Transformation Formulas

Put $z = -1$ in 3.12(2) and use 3.13.1(1). Then

$${}_2F_1(2a, 2a - c + 1; c; -1) = \frac{2^{-2a}(c-a)_a}{(\frac{1}{2})_a}, \qquad c \neq 0, -1, -2, \ldots . \tag{1}$$

Similarly with $z = \frac{1}{2}$ in 3.12(4) and 3.12(6) we get, respectively,

$${}_2F_1(2a, 2b; a + b + \tfrac{1}{2}; \tfrac{1}{2}) = \frac{\pi^{1/2}\Gamma(a+b+\frac{1}{2})}{\Gamma(a+\frac{1}{2})\,\Gamma(b+\frac{1}{2})},$$
$$a + b + \tfrac{1}{2} \neq 0, -1, -2, \ldots, \tag{2}$$

$${}_2F_1(a, 1-a; c; \tfrac{1}{2}) = \frac{2^{1-c}\pi^{1/2}\Gamma(c)}{\Gamma[(c+a)/2]\,\Gamma[(c-a+1)/2]},$$
$$c \neq 0, -1, -2, \ldots . \tag{3}$$

From 3.1(23),

$${}_2F_1\left(\begin{matrix}-n, c\\ 2c\end{matrix}\middle|\, 2\right) = \frac{(-)^n 2^n (c)_n}{(2c)_n}\, {}_2F_1\left(\begin{matrix}-n, 1-n-2c\\ 1-n-c\end{matrix}\middle|\, \frac{1}{2}\right), \qquad c \neq 1, 2, \ldots, \tag{4}$$

and with the aid of (2), the latter is nil if n is odd, while

$${}_2F_1\left(\begin{matrix}-2n, c\\ 2c\end{matrix}\middle|\, 2\right) = \frac{(\frac{1}{2})_n}{(c+\frac{1}{2})_n}, \qquad c \neq 0, -1, -2, \ldots . \tag{5}$$

Similar results for f, g, g_1, and G follow from the results given in 3.12. These are presented below without proof. See Nörlund (1963) for details.

$$f(2a, 2b; c; \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2}-a)\,\Gamma(\frac{1}{2}-b)}{\pi^{1/2}\Gamma(1-c)}, \tag{6}$$

where $c = a + b + \frac{1}{2}$ is a negative integer, a or $b = 0, -1, \ldots, -[|c|/2]$,

$$f(2a, 2b, c, \tfrac{1}{2}) = 0, \tag{7}$$

where c is as in (6), a or $b = -\frac{1}{2}, -\frac{3}{2}, \ldots, \frac{1}{2} + [\frac{1}{2}c]$

$$f(2a, 2a - c + 1, c, -1) = \frac{2^{-2a-c}(1-c)_a}{(\frac{1}{2} - a)_a}, \tag{8}$$

where c is a negative integer, a is a nonpositive integer $\geqslant c/2$, $\epsilon = 0$, $2a = c - 1, c - 2, \ldots, 2c - 1$, $\epsilon = 1$,

$$f(a, 1 - a, c, \tfrac{1}{2}) = \frac{\Gamma(1 - (a + c)/2)\,\Gamma((1 + a - c)/2)}{2^c \pi^{1/2} \Gamma(1 - c)}, \tag{9}$$

where a is an integer, $c \leqslant a \leqslant 1 - c$, c a negative integer

In the following formulas, $c = s + 1$, s a positive integer or zero

$$g(2a, 2b, c, \tfrac{1}{2}) = \frac{(-)^{s+1} s!\, \pi^{3/2}}{\Gamma(\frac{1}{2} + a)\,\Gamma(\frac{1}{2} + b) \cos \pi(a - b)}, \tag{10}$$

where $c = a + b + \frac{1}{2}$, and neither $2a$ nor $2b$ is an integer

$$G(2a, 2b, c, \tfrac{1}{2}) = \frac{s!\, \pi^{1/2}}{\Gamma(a + \frac{1}{2})\,\Gamma(b + \frac{1}{2})} \Big[\psi(1 + s) + \psi(1) - \psi(1 - a) - \psi(1 - b) - 2 \ln 2 + \sum_{k=1}^{2s} \frac{(-)^k}{2a - k}\Big],$$

$$c = a + b + \tfrac{1}{2}, \qquad 2a \neq 1, 2, \ldots, 2s \tag{11}$$

$$\begin{aligned} G(2a, 2a - c + 1, c, -1 \pm i0) &= \frac{2^{-2a} \pi^{1/2} s!}{\Gamma(a + \frac{1}{2})\,\Gamma(s + 1 - a)} [\pm i\pi - 2 \ln 2 + \psi(1 + s) + \psi(1) \\ &\quad - \psi(1 - a) - \psi(\tfrac{1}{2} + a) \\ &\quad - \psi(s + 1 - 2a) + \psi(1 - 2a)], \end{aligned}$$

$$2a \neq 1, 2, \ldots, 2s \tag{12}$$

$$g(2a, 2a - c + 1, c, -1 \pm i0) = \frac{(-)^{s+1} 2^{-2a} s!\, \pi^{1/2} \Gamma(a - s)}{\Gamma(a + \frac{1}{2})} e^{\pm i a \pi}, \tag{13}$$

where $2a$ is not an integer $< 2s + 1$

$$g(2a, 2a - c + 1, c, -1 + i0) - g(2a, 2a - c + 1, c, -1 - i0) = \frac{(2\pi i)\, s!\, \pi^{1/2} 2^{-2a}}{\Gamma(a + \frac{1}{2})\,\Gamma(1 + s - a)}, \tag{14}$$

and this formula is also valid if g is replaced by G.

$$G(a, 1-a; c; \tfrac{1}{2}) = \frac{s!\,\pi^{1/2}2^{-s}}{\Gamma[(s+2-a)/2]\,\Gamma[(s+1+a)/2]}$$
$$\times [\psi(1+s) + \psi(1) - 2\psi(a) + \pi \tan \pi[(s+1+a)/2]\,,$$
$$a \neq 0, \pm 1, \pm 2, \ldots, \pm(s-1), s. \tag{15}$$

$$g(a, 1-a; c; \tfrac{1}{2}) = (-)^{s+1} s!\,\pi^{-1/2}2^{-s-1}\Gamma[(1-s-a)/2]\,\Gamma[(a-s)/2],$$
$$a \text{ is not an integer.} \tag{16}$$

$$g_1(a, 1-a; c; \tfrac{1}{2} \pm i0) = \frac{s!\,\pi^{1/2}2^{-s}\Gamma[(1-s-a)/2]}{\Gamma[(s+2-a)/2]} \exp\left[\mp \frac{i\pi}{2}(a+s+1)\right],$$
$$(a+s) \text{ is not a positive integer.} \tag{17}$$

3.13.3. Special ${}_{p+1}F_p$'s, $p > 1$ of Unit Argument

Write 3.8(2) in the form

$$(1-z)^{a+b-c}\,{}_2F_1(a, b; c; z) = {}_2F_1(c-a, c-b; c; z), \tag{1}$$

expand both sides in powers of z, and equate like coefficients of z^n. We get Saalschütz's formula

$${}_3F_2\left(\begin{matrix} -n, a, b \\ c, 1+a+b-c-n \end{matrix}\middle|\, 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n},$$
$$n \text{ a positive integer or zero.} \tag{2}$$

We also have

$${}_3F_2\left(\begin{matrix} -n, n+a, b \\ c, a+b+1-c \end{matrix}\middle|\, 1\right) = \frac{(c-b)_n(a-c+1)_n}{(c)_n(a+b+1-c)_n}. \tag{3}$$

If in ${}_{p+1}F_p(a_{p+1}\,;\, b_p\,;\, z)$, the parameters are such that

$$1 + \sum_{j=1}^{p+1} a_j = \sum_{j=1}^{p} b_j\,, \tag{4}$$

then the ${}_{p+1}F_p$ is said to be Saalschützian. Thus the series in (2) and (3) are Saalschützian of the terminating type. If

$$1 + a_1 = b_1 + a_2 = \cdots = b_p + a_{p+1}\,, \tag{5}$$

then the series is said to be well poised, and if all but one of these equalities are true, the series is said to be nearly poised. By Dixon's

theorem, see Bailey (1935) or Slater (1966), every well-poised ${}_3F_2$ of unit argument can be summed Thus,

$$
{}_3F_2\left(\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix}\middle| 1\right) = \frac{\Gamma(1+\frac{1}{2}a)\,\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma(1+\frac{1}{2}a-b)} \times \frac{\Gamma(1+a-c)\,\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+\frac{1}{2}a-c)\,\Gamma(1+a-b-c)},
$$

$$
R(a-2b-2c) > -2 \qquad (6)
$$

Two other theorems which sum a ${}_3F_2$ of unit argument are

$$
{}_3F_2\left(\begin{matrix} a, b, c \\ \frac{1}{2}(a+b+1), 2c \end{matrix}\middle| 1\right) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+c)\,\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\,\Gamma(\frac{1}{2}+\frac{1}{2}b)} \times \frac{\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\,\Gamma(\frac{1}{2}-\frac{1}{2}b+c)},
$$

$$
R(2c-a-b) > -1 \qquad (7)
$$

$$
{}_3F_2\left(\begin{matrix} a, 1-a, c \\ f, 2c+1-f \end{matrix}\middle| 1\right) = \frac{\pi\Gamma(f)\,\Gamma(2c+1-f)}{2^{2c-1}\Gamma[c+\frac{1}{2}(a+1-f)]\,\Gamma[\frac{1}{2}(a+f)]\,\Gamma[1+c-\frac{1}{2}(a+f)]\,\Gamma[\frac{1}{2}(1-a+f)]},
$$

$$
R(c) > 0 \qquad (8)
$$

Equations (7) and (8) are due to Watson and Whipple, respectively [see Bailey (1935)] Now the left-hand side of (8) is meaningful if c is a negative integer or zero In this event its value is not given by the right-hand side of (8) unless a is an integer or zero as pointed out by Dzrbasjan (1964) who shows that

$$
{}_3F_2\left(\begin{matrix} -n, a, 1-a \\ f, -2n-f+1 \end{matrix}\middle| 1\right) = \frac{2^{2n}(\frac{1}{2}a+\frac{1}{2}f)_n(\frac{1}{2}a-\frac{1}{2}f+\frac{1}{2}-n)_n}{(f)_n(1-2n-f)_n} \qquad (9)
$$

The following account of transformations of ${}_3F_2$ series of unit argument is adapted from Bailey (1935), see also Slater (1966) The first formula is

$$
{}_3F_2\left(\begin{matrix} a, b, c \\ e, f \end{matrix}\middle| 1\right) = \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(s)}{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)}\,{}_3F_2\left(\begin{matrix} e-a, f-a, s \\ s+b, s+c \end{matrix}\middle| 1\right),
$$

$$
s = e+f-a-b-c, \qquad s \neq 0 \qquad (10)
$$

A second important relation is

$$
{}_3F_2\left(\begin{matrix} a, b, c \\ e, f \end{matrix}\middle| 1\right) = \frac{\Gamma(1-a)\,\Gamma(e)\,\Gamma(f)\,\Gamma(c-b)}{\Gamma(e-b)\,\Gamma(f-b)\,\Gamma(1+b-a)\,\Gamma(c)} \times {}_3F_2\left(\begin{matrix} b, b-e+1, b-f+1 \\ 1+b-c, 1+b-a \end{matrix}\middle| 1\right)
$$

$$
+ \text{ a similar expression with } b \text{ and } c \text{ interchanged} \qquad (11)
$$

These are two of many relations involving ${}_3F_2$'s of unit argument. Let r_i, $i = 0(1)5$, be parameters such that

$$\sum_{i=0}^{5} r_i = 0. \tag{12}$$

Let

$$\alpha_{lmn} = \tfrac{1}{2} + r_l + r_m + r_n, \qquad \beta_{mn} = 1 + r_m - r_n, \tag{13}$$

$$F_p(u; v, w) = [\Gamma(\alpha_{xyz})\,\Gamma(\beta_{vu})\,\Gamma(\beta_{wu})]^{-1}\,{}_3F_2\left(\begin{matrix}\alpha_{vwx}, \alpha_{vwy}, \alpha_{vwz}\\ \beta_{vu}, \beta_{wu}\end{matrix}\,\middle|\,1\right), \tag{14}$$

$$F_n(u; v, w) = [\Gamma(\alpha_{uvw})\,\Gamma(\beta_{uv})\,\Gamma(\beta_{uw})]^{-1}\,{}_3F_2\left(\begin{matrix}\alpha_{uyz}, \alpha_{uzx}, \alpha_{uxy}\\ \beta_{uv}, \beta_{uw}\end{matrix}\,\middle|\,1\right), \tag{15}$$

where the subscripts u, v, w, x, y, and z are distinct and all must take on one of the numbers 0, 1,..., 5, and where the α's and β's are defined in terms of the parameters a, b, c, d, e, f, and $s = e + f - a - b - c$ in Tables 3.3–3.5. Observe that (14), (15) are symmetric in x, y, and z and also in v and w. It may be shown that

$$F_p(u; v_1, w_1) = F_p(u; v_2, w_2), \tag{16}$$

$$F_n(u; v_1, w_1) = F_n(u; v_2, w_2), \tag{17}$$

and the condition for convergence of (16) and (17) is $R(\alpha_{xyz}) > 0$ and $R(\alpha_{uvw}) > 0$, respectively. Observe that F_n comes from the corresponding F_p by changing the sign of all the r_i's.

TABLE 3.3[a]

EXPRESSIONS FOR α'S AND β'S IN TERMS OF a, b, c, e, f $(s = e + f - a - b - c)$

α		β		
$\alpha_{012} = 1-c$	$\alpha_{123} = s$	$\beta_{01} = 2-s-a$	$\beta_{20} = s+b$	$\beta_{40} = e$
$\alpha_{013} = 1-b$	$\alpha_{124} = e-c$	$\beta_{02} = 2-s-b$	$\beta_{21} = 1-a+b$	$\beta_{41} = 1+b+c-f$
$\alpha_{014} = 1-f+a$	$\alpha_{125} = f-c$	$\beta_{03} = 2-s-c$	$\beta_{23} = 1+b-c$	$\beta_{42} = 1+a+c-f$
$\alpha_{015} = 1-e+a$	$\alpha_{134} = e-b$	$\beta_{04} = 2-e$	$\beta_{24} = 1-a-c+f$	$\beta_{43} = 1+a+b-f$
$\alpha_{023} = 1-a$	$\alpha_{135} = f-b$	$\beta_{05} = 2-f$	$\beta_{25} = 1-a-c+e$	$\beta_{45} = 1+e-f$
$\alpha_{024} = 1-f+b$	$\alpha_{145} = a$	$\beta_{10} = s+a$	$\beta_{30} = s+c$	$\beta_{50} = f$
$\alpha_{025} = 1-e+b$	$\alpha_{234} = e-a$	$\beta_{12} = 1+a-b$	$\beta_{31} = 1+c-a$	$\beta_{51} = 1+b+c-e$
$\alpha_{034} = 1-f+c$	$\alpha_{235} = f-a$	$\beta_{13} = 1+a-c$	$\beta_{32} = 1-b+c$	$\beta_{52} = 1+a+c-e$
$\alpha_{035} = 1-e+c$	$\alpha_{245} = b$	$\beta_{14} = 1-b-c+f$	$\beta_{34} = 1-a-b+f$	$\beta_{53} = 1+a+b-e$
$\alpha_{045} = 1-s$	$\alpha_{345} = c$	$\beta_{15} = 1-b-c+e$	$\beta_{35} = 1-a-b+e$	$\beta_{54} = 1-e+f$

[a] From W. N. Bailey, "Generalized Hypergeometric Series." Cambridge Univ. Press, London and New York, 1935. Reprinted with permission.

TABLE 3.4*

$$F_v(u\ v\ w) = \frac{1}{\Gamma(\alpha_{uvw})\,\Gamma(\beta_{uv})\,\Gamma(\beta_{uw})}\,{}_3F_2\left(\begin{matrix}\alpha_{uvw} & \alpha_{vwu} & \alpha_{wuv}\\ \beta_{uv} & \beta_{uw} & \end{matrix}\middle|\,1\right)$$

	$v\ w$	Numerator parameters			Denominator parameters	
$F_v(0)$	4 5	a	b	c	e	f
	2 3	s	$e-a$	$f-a$	$s+b$	$s+c$
	1 4	a	$e-b$	$e-c$	e	$s+a$
$F_v(1)$ [$F_v(2)$ and $F_v(3)$ are of this type]	0 2	$1-e+b$	$1-f+b$	$1-a$	$1+b-a$	$2-s-a$
	0 4	$1-s$	$1-f+b$	$1-f+c$	$1+b+c-f$	$2-s-a$
	2, 3	$e-a$	$f-a$	$1-a$	$1-a+b$	$1-a+c$
	2 4	b	$e-a$	$1-f+b$	$1+b+c-f$	$1+b-a$
	4 5	$1-s$	b	c	$1+b+c-f$	$1+b+c-e$
$F_v(4)$ [$F_v(5)$ is of this type]	0 1	$1-e+a$	$1-b$	$1-c$	$2-e$	$1+f-b-c$
	0 5	$1-e+a$	$1-e+b$	$1-e+c$	$2-s$	$1-e+f$
	1 2	$f-c$	$1-c$	s	$1+f-a-c$	$1+f-b-c$
	1 5	$1-e+a$	$f-e$	$f-b$	$1+f-e$	$1+f-b-c$

* From W. N. Bailey, Generalized Hypergeometric Series, Cambridge Univ. Press, London and New York, 1935. Reprinted with permission.

TABLE 3.5*

$$F_u(u\ v\ w) = \frac{1}{\Gamma(\alpha_{uvw})\,\Gamma(\beta_{uv})\,\Gamma(\beta_{uw})}\,{}_3F_2\left(\begin{matrix}\alpha_{uvw} & \alpha_{uwv} & \alpha_{uvw}\\ \beta_{uv} & \beta_{uw} & \end{matrix}\middle|\,1\right)$$

	$v\ w$	Numerator parameters			Denominator parameters	
$F_u(0)$	4 5	$1-a$	$1-b$	$1-c$	$2-e$	$2-f$
	2, 3	$1-s$	$1-e+a$	$1-f+a$	$2-s-b$	$2-s-c$
	1 4	$1-a$	$1-e+b$	$1-e+c$	$2-e$	$2-s-a$
$F_u(1)$ [$F_u(2)$ and $F_u(3)$ are of this type]	0 2	$e-b$	$f-b$	a	$1+a-b$	$s+a$
	0 4	s	$f-b$	$f-c$	$1-b-c+f$	$s+a$
	2 3	$1-e+a$	$1-f+a$	a	$1+a-b$	$1+a-c$
	2, 4	$1-b$	$1-e+a$	$f-b$	$1-b-c+f$	$1+a-b$
	4 5	s	$1-b$	$1-c$	$1-b-c+f$	$1-b-c+e$
$F_u(4)$ [$F_u(5)$ is of this type]	0 1	$e-a$	b	c	e	$1-f+b+c$
	0 5	$e-a$	$e-b$	$e-c$	e	$1+e-f$
	1 2	$1-f+c$	c	$1-s$	$1-f+a+c$	$1-f+b+c$
	1 5	$e-a$	$1-f+c$	$1-f+b$	$1+e-f$	$1-f+b+c$

* From W. B. Bailey, Generalized Hypergeometric Series, Cambridge Univ. Press, London and New York, 1935. Reprinted with permission.

Thus (10) is the same as

$$F_p(0; 2, 3) = F_p(0; 4, 5) \tag{18}$$

and (14) states that all ten of the functions $F_p(0; v, w)$, $v, w = 1, 2, \ldots, 5$, $v \neq w$, are equal to each other. We therefore refer to these functions as simply $F_p(0)$. Likewise, the statement

$$F_n(2; 1, 5) = F_n(2; 3, 4) \tag{19}$$

means that

$$\begin{aligned} {}_3F_2\left(\begin{matrix} e-a, 1-f+b, 1-a \\ 1-a+b, 1-a-c+e \end{matrix}\middle| 1\right) &= \frac{\Gamma(f-c)\,\Gamma(1-a+b)\,\Gamma(1-a-c+e)}{\Gamma(e-a)\,\Gamma(1-c+b)\,\Gamma(1-a-c+f)} \\ &\quad \times {}_3F_2\left(\begin{matrix} f-c, 1-e+b, 1-c \\ 1-c+b, 1-a-c+f \end{matrix}\middle| 1\right), \end{aligned} \tag{20}$$

and by a change of notation this is equivalent to (10). We also have that all ten of the functions $F_n(2; v, w)$, $v, w = 0, 1, 3, 4, 5$, $v \neq w$, are equal to each other, and we refer to these functions as $F_n(2)$.

We next consider the relation (11). In our present notation this becomes

$$\frac{\sin \pi\beta_{23}}{\pi\Gamma(\alpha_{023})} F_p(0) = \frac{F_n(2)}{\Gamma(\alpha_{134})\,\Gamma(\alpha_{135})\,\Gamma(\alpha_{345})} - \frac{F_n(3)}{\Gamma(\alpha_{124})\,\Gamma(\alpha_{125})\,\Gamma(\alpha_{245})}. \tag{21}$$

Similarly, by changing the signs of the r_i terms,

$$\frac{\sin \pi\beta_{32}}{\pi\Gamma(\alpha_{145})} F_n(0) = \frac{F_p(2)}{\Gamma(\alpha_{025})\,\Gamma(\alpha_{024})\,\Gamma(\alpha_{012})} - \frac{F_p(3)}{\Gamma(\alpha_{035})\,\Gamma(\alpha_{034})\,\Gamma(\alpha_{013})}. \tag{22}$$

Upon combining three equations like (21), we get

$$\begin{aligned} \frac{\sin \pi\beta_{45}}{\Gamma(\alpha_{012})\,\Gamma(\alpha_{013})\,\Gamma(\alpha_{023})} F_p(0) &+ \frac{\sin \pi\beta_{50}}{\Gamma(\alpha_{124})\,\Gamma(\alpha_{134})\,\Gamma(\alpha_{234})} F_p(4) \\ &+ \frac{\sin \pi\beta_{04}}{\Gamma(\alpha_{125})\,\Gamma(\alpha_{135})\,\Gamma(\alpha_{235})} F_p(5) = 0, \end{aligned} \tag{23}$$

and with a change of notation, we have

$$\begin{aligned} \frac{\sin \pi\beta_{54}}{\Gamma(\alpha_{345})\,\Gamma(\alpha_{245})\,\Gamma(\alpha_{145})} F_n(0) &+ \frac{\sin \pi\beta_{05}}{\Gamma(\alpha_{035})\,\Gamma(\alpha_{025})\,\Gamma(\alpha_{015})} F_n(4) \\ &+ \frac{\sin \pi\beta_{40}}{\Gamma(\alpha_{034})\,\Gamma(\alpha_{024})\,\Gamma(\alpha_{014})} F_n(5) = 0. \end{aligned} \tag{24}$$

Now eliminate $F_n(2)$ from the equation of the type (21) which connects $F_p(5)$, $F_n(0)$, and $F_n(2)$, and from the equation of type (22) which connects $F_n(2)$, $F_p(0)$, and $F_p(5)$ Then

$$\frac{F_p(0)}{\Gamma(\alpha_{120})\,\Gamma(\alpha_{130})\,\Gamma(\alpha_{230})\,\Gamma(\alpha_{240})\,\Gamma(\alpha_{140})\,\Gamma(\alpha_{340})}$$

$$+\frac{\sin \pi\beta_{05} F_n(0)}{\pi\Gamma(\alpha_{123})\,\Gamma(\alpha_{124})\,\Gamma(\alpha_{134})\,\Gamma(\alpha_{234})} = K_0 F_p(5)$$

$$\pi^3 K_0 = (\sin \pi\alpha_{145})(\sin \pi\alpha_{245})(\sin \pi\alpha_{345})$$

$$+ (\sin \pi\alpha_{123})(\sin \pi\beta_{40})(\sin \pi\beta_{50}), \quad (25)$$

and likewise

$$\frac{F_n(0)}{\Gamma(\alpha_{345})\,\Gamma(\alpha_{245})\,\Gamma(\alpha_{145})\,\Gamma(\alpha_{135})\,\Gamma(\alpha_{235})\,\Gamma(\alpha_{125})}$$

$$+\frac{\sin \pi\beta_{50} F_p(0)}{\pi\Gamma(\alpha_{045})\,\Gamma(\alpha_{035})\,\Gamma(\alpha_{025})\,\Gamma\alpha(_{015})} = K_0 F_n(5) \quad (26)$$

All the three-term relations between the 120 hypergeometric series are typified by (21)–(26)

To further illustrate use of the tables, consider

$$ {}_3F_2\left(\begin{matrix} a & b & c \\ e & f \end{matrix}\middle| 1\right) = \frac{\Gamma(e)\,\Gamma(e-a-b)}{\Gamma(e-a)\,\Gamma(e-b)}\, {}_3F_2\left(\begin{matrix} a, b, f-c \\ a+b-e+1, f \end{matrix}\middle| 1\right)$$

$$+\frac{\Gamma(e)\,\Gamma(f)\,\Gamma(a+b-e)\,\Gamma(e+f-a-b-c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(f-c)\,\Gamma(e+f-a-b)}$$

$$\times\, {}_3F_2\left(\begin{matrix} e-a, e-b, e+f-a-b-c \\ e-a-b+1\ \ e+f-a-b \end{matrix}\middle| 1\right) \quad (27)$$

This is equivalent to

$$F_p(0, 4\ \ 5) = \frac{\pi}{\sin \pi\beta_{53}}\,\frac{\Gamma(\alpha_{005})}{\Gamma(\alpha_{234})\,\Gamma(\alpha_{134})\,\Gamma(\alpha_{123})} F_n(5, 0, 3)$$

$$+\frac{\pi}{\sin \pi\beta_{35}}\,\frac{\Gamma(\alpha_{035})}{\Gamma(\alpha_{145})\,\Gamma(\alpha_{245})\,\Gamma(\alpha_{145})} F_n(3, 0\ \ 5), \quad (28)$$

which in turn is the same as (21) if there we interchange the indices 2 and 5

Suppose that in (21), $e + f = a + b + c + 1$ Then the ${}_3F_2$ on the left is Saalschutzian and the first series on the right becomes a ${}_3F_1$, which can be summed

Thus,

$$
\begin{aligned}
{}_3F_2\left(\begin{matrix} a, b, e+f-a-b-1 \\ e, f \end{matrix}\middle| 1\right) &= \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(e-a-b)\,\Gamma(f-a-b)}{\Gamma(e-a)\,\Gamma(e-b)\,\Gamma(f-a)\,\Gamma(f-b)} \\
&\quad + \frac{\Gamma(e)\,\Gamma(f)}{(a+b-e)\,\Gamma(a)\,\Gamma(b)\,\Gamma(e+f-a-b)} \\
&\quad \times {}_3F_2\left(\begin{matrix} e-a, e-b, 1 \\ e-a-b+1, e+f-a-b \end{matrix}\middle| 1\right), \\
&\qquad e+f = a+b+c+1. \qquad (29)
\end{aligned}
$$

If a or b is a negative integer, the second term on the right-hand side of this equation is nil and we get Saalschütz's formula (2).

If c in ${}_3F_2(a, b, c; e, f; 1)$ is a negative integer, say $c = -m$, we can reverse the series [see 3.2(3)] and with $\alpha_{345} = -c = m$, we have

$$\Gamma(\alpha_{124})\,\Gamma(\alpha_{125})\,F_p(0) = (-)^m\Gamma(\alpha_{023})\,\Gamma(\alpha_{013})\,F_n(3). \qquad (30)$$

This is a degenerate form of (21). In all there are eighteen terminating series. Three of these are $F_p(0; 4, 5)$, $F_p(0; 3, 5)$, and $F_p(0; 3, 4)$, which when reversed give $F_n(3; 1, 2)$, $F_n(4; 1, 2)$, and $F_n(5; 1, 2)$, respectively. The relations between the eighteen series are described by

$$
\begin{aligned}
&\Gamma(\alpha_{123})\,\Gamma(\alpha_{124})\,\Gamma(\alpha_{125})\,F_p(0) \\
&\quad = \Gamma(\alpha_{0g3})\,\Gamma(\alpha_{0g4})\,\Gamma(\alpha_{0g5})\,F_p(3-g), \qquad g = 1, 2, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma(\alpha_{123})\,\Gamma(\alpha_{124})\,\Gamma(\alpha_{125})\,F_p(0) \\
&\quad = (-)^m\Gamma(\alpha_{12h})\,\Gamma(\alpha_{02h})\,\Gamma(\alpha_{01h})\,F_n(h), \qquad h = 3, 4, 5. \qquad (32)
\end{aligned}
$$

The other series, such as $F_n(0)$, do not give any specially simple formulas. For an application of formulas (11), (14), (16), (31), see the discussion surrounding 8.2(60–68).

Some miscellaneous results now follow. From 3.2(7),

$$y_m\left(\begin{matrix} a, b \\ c \end{matrix}\middle| 1\right) = \sum_{k=0}^{m} \frac{(a)_k(b)_k}{(c)_k k!} = \frac{(a)_m(b)_m}{(c)_m m!}\,{}_3F_2\left(\begin{matrix} -m, 1-m-c, 1 \\ 1-m-a, 1-m-b \end{matrix}\middle| 1\right). \qquad (33)$$

If $c = a + b + 1$, the ${}_3F_2$ is Saalschützian and so may be summed by (2). If $c = a + b$, $y_m \to \infty$ as $m \to \infty$ in view of 3.13.1(2). We now prove that

$$
\begin{aligned}
y_m\left(\begin{matrix} a, b \\ a+b \end{matrix}\middle| 1\right) &= \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\Big\{\psi(b+m+1) + \psi(1) - \psi(a) - \psi(b) \\
&\quad - \frac{b(1-a)}{(b+m+1)}\,{}_4F_3\left(\begin{matrix} b+1, 2-a, 1, 1 \\ b+m+2, 2, 2 \end{matrix}\middle| 1\right)\Big\}, \qquad (34)
\end{aligned}
$$

and so from 2 11(8),

$$y_m\left(\begin{matrix}a, b\\ a+b\end{matrix}\middle|\,1\right) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\Big[\ln(b+m+1) + \psi(1) - \psi(b) - \psi(a)$$
$$- \frac{\frac{1}{2} + b(1-a)}{b+m+1}$$
$$- \frac{\{3b(b+1)(1-a)(2-a)+1\}}{12(b+m+1)^2} + O((b+m+1)^{-3})\Big] \tag{35}$$

Let $\epsilon > 0$ Then

$$y_m\left(\begin{matrix}a, b\\ a+b+\epsilon\end{matrix}\middle|\,1\right) = {}_2F_1\left(\begin{matrix}a, b\\ a+b+\epsilon\end{matrix}\middle|\,1\right) - \frac{(a)_{m+1}(b)_{m+1}}{(a+b+\epsilon)_{m+1}(1)_{m+1}}$$
$$\times\, {}_3F_2\left(\begin{matrix}a+m+1, b+m+1, 1\\ a+b+\epsilon+m+1, m+2\end{matrix}\middle|\,1\right)$$

and using 3 13 1(1) and (10), we obtain

$$y_m\left(\begin{matrix}a, b\\ a+b+\epsilon\end{matrix}\middle|\,1\right) = \frac{\Gamma(a+b+\epsilon)\,\Gamma(\epsilon+1)}{\epsilon}\Big[\frac{1}{\Gamma(a+\epsilon)\,\Gamma(b+\epsilon)}$$
$$- \frac{(b)_{m+1}}{\Gamma(a)\,\Gamma(b+m+1+\epsilon)\,\Gamma(\epsilon+1)}$$
$$\times\, {}_3F_2\left(\begin{matrix}b+\epsilon, 1-a, \epsilon\\ b+m+1+\epsilon, 1+\epsilon\end{matrix}\middle|\,1\right)\Big]$$

Next split off the first term in the expansion of the latter ${}_3F_2$ and let $\epsilon \to 0$ Then (34) follows

If in a ${}_3F_2$ form, a numerator parameter exceeds a denominator parameter by a positive integer, say m, the ${}_3F_2$ may be expressed as the sum of $(m+1)$ ${}_2F_1$'s If the latter can be summed, then so can the ${}_3F_2$ In illustration, since

$${}_3F_2\left(\begin{matrix}a, b\ c+1\\ d\ c\end{matrix}\middle|\,z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c+k)\,z^k}{(d)_k k!\,c} = {}_2F_1\left(\begin{matrix}a\ b\\ d\end{matrix}\middle|\,z\right)$$
$$+ \frac{abz}{cd}\,{}_2F_1\left(\begin{matrix}a+1, b+1\\ d+1\end{matrix}\middle|\,z\right), \tag{36}$$

$${}_3F_2\left(\begin{matrix}a, b\ c+1\\ d, c\end{matrix}\middle|\,1\right) = \frac{\Gamma(d)\,\Gamma(d-a-b)}{\Gamma(d-a)\,\Gamma(d-b)}\left[1 - \frac{ab}{c(a+b+1-d)}\right],$$
$$R(d-a-b) > 1 \tag{37}$$

This is a special case of 9 1(34)

The formula

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ d+1, c+1 \end{matrix}\middle| z\right) = \frac{c}{c-d}\,{}_2F_1\left(\begin{matrix} a, b \\ d+1 \end{matrix}\middle| z\right) - \frac{d}{c-d}\,{}_3F_2\left(\begin{matrix} a, b, c \\ d, c+1 \end{matrix}\middle| z\right), \tag{38} $$

can be used to prove

$$ {}_3F_2\left(\begin{matrix} -n, n+2\nu, \mu+\nu+1 \\ 2\nu+1, \mu+\nu+2 \end{matrix}\middle| 1\right) = \frac{(2\nu)\,n!\,(\nu-\mu-1)_n}{(\nu-\mu-1)(2\nu)_n(\mu+\nu+2)_n}, \quad n > 0. \tag{39} $$

By integration of 3.1(13) and 3.6(1), or otherwise, we get

$$ z\,{}_3F_2\left(\begin{matrix} a, b, 1 \\ c, 2 \end{matrix}\middle| z\right) = \frac{(c-1)}{(a-1)(b-1)}\left[{}_2F_1\left(\begin{matrix} a-1, b-1 \\ c-1 \end{matrix}\middle| z\right) - 1\right], \tag{40} $$

and so

$$ {}_3F_2\left(\begin{matrix} a, b, 1 \\ c, 2 \end{matrix}\middle| 1\right) = \frac{(c-1)}{(a-1)(b-1)}\left[\frac{\Gamma(c-1)\,\Gamma(c-a-b+1)}{\Gamma(c-a)\,\Gamma(c-b)} - 1\right], $$
$$ a \neq 1, \qquad b \neq 1, \qquad R(c-a-b) > -1. \tag{41} $$

By L'Hospital's theorem,

$$ {}_3F_2\left(\begin{matrix} a, 1, 1 \\ c, 2 \end{matrix}\middle| 1\right) = \frac{(c-1)}{(a-1)}\,[\psi(c-1) - \psi(c-a)],\ a \neq 1,\ R(c-a) > 0, \tag{42} $$

$$ {}_3F_2\left(\begin{matrix} 1, 1, 1 \\ c, 2 \end{matrix}\middle| 1\right) = (c-1)\,\psi'(c-1), \qquad R(c) > 1. \tag{43} $$

Again, integration of (40) leads to

$$ z^2\,{}_3F_2\left(\begin{matrix} a, b, 1 \\ c, 3 \end{matrix}\middle| z\right) = \frac{2(c-2)_2}{(a-2)_2(b-2)_2}\left[{}_2F_1\left(\begin{matrix} a-2, b-2 \\ c-2 \end{matrix}\middle| z\right) - 1\right] - \frac{2z(c-1)}{(a-1)(b-1)}, \tag{44} $$

which in turn yields

$$ {}_3F_2\left(\begin{matrix} a, b, 1 \\ c, 3 \end{matrix}\middle| 1\right) = \frac{2(c-2)_2}{(a-2)_2(b-2)_2}\left[\frac{\Gamma(c-2)\,\Gamma(c-a-b+2)}{\Gamma(c-a)\,\Gamma(c-b)} - 1\right] - \frac{2(c-1)}{(a-1)(b-1)}, $$
$$ a \neq 1, 2; \qquad b \neq 1, 2; \qquad R(c-a-b) > -2. \tag{45} $$

Results for $a = 1, 2$, etc, can be found by use of L'Hospital's theorem

Next we show how to evaluate a certain terminating ${}_4F_3$ of unit argument The technique seems novel in this context, and we develop the process in some detail The result is due to J L Fields and Y L Luke We prove that

$$ {}_4F_3\left(\begin{matrix}-n, f+1, 1, \dfrac{f(n+e-1)+z(e-1-f)}{n+f} \\ e, f, 1-z\end{matrix}\middle| 1\right) = \frac{z(n+f)(e-1)}{f(z-n)(n+e-1)}, $$

$$ n \text{ a positive integer or zero}, \qquad f(z-n)(n+e-1) \neq 0 \tag{46} $$

For particular values of the parameters in (46), limiting forms of the left hand side must be taken The same is true in what follows Consider

$$ V(z) = (-z)^{-1}\, {}_4F_3\left(\begin{matrix}-n, b, c, d+hz \\ e, f, 1-z\end{matrix}\middle| 1\right) \tag{47} $$

Clearly $V(z)$ is a rational function of z with poles at $z = 0, 1, \ldots, n$ To develop the partial fraction decomposition of $V(z)$ we proceed as follows If

$$ y_k(z) = \frac{(d+hz)_k}{-z(1-z)_k}, \qquad k = 0, 1, \ldots, n $$

then by residue theory

$$ y_k(z) = \sum_{r=0}^{k} \frac{(-)^r (d+hr)_k}{(r-z)\, r!\, (k-r)!}, $$

so that

$$ V(z) = \sum_{k=0}^{n} \frac{(-n)_k (b)_k (c)_k}{(e)_k (f)_k k!} \sum_{r=0}^{k} \frac{(-)^r (d+hr)_k}{(r-z)\, r!\, (k-r)!} $$

Upon interchanging the order of summations, we find

$$ \begin{aligned} V(z) &= \sum_{r=0}^{n} \frac{(-)^r (-n)_r (b)_r (c)_r (d+hr)_r}{(r-z)(e)_r (f)_r (r!)^2}\, q_r \\ q_r &= {}_4F_3\left(\begin{matrix}r-n, r+b, r+c, d+(h+1)r \\ r+e, r+f, r+1\end{matrix}\middle| 1\right) \end{aligned} \tag{48} $$

Next we specialize the parameters so as to obtain a particularly simple form for q_r Thus let $c = 1$, and $b = f + 1$ Then from (37), we find

$$ \begin{aligned} q_r &= \frac{\Gamma(r+e)\,\Gamma(n+e-d-r(h+1))}{\Gamma(n+e)\,\Gamma(e-d-rh)} \\ &\quad \times \left[1 - \frac{(r-n)(d+r(h+1))}{(r+f)(-n+d+1-e+r(h+1))}\right] \end{aligned} $$

Now set

$$d + r(h+1) = \rho(r+f), \qquad -n + d + 1 - e + r(h+1) = \rho(r-n),$$
$$\rho = h+1, \qquad r = 1, 2, \ldots, n-1.$$

Thus with

$$h = (e - f - 1)/(n+f), \qquad d = f(n+e-1)/(n+f),$$

we have

$$q_r = 0, \qquad r = 0, 1, \ldots, n-1, \quad q_n = 1,$$

and (46) readily follows. The following special cases are worth noting:

$${}_4F_3\left(\begin{matrix}-n, \beta+1, 1, z+2\beta \\ n+2\beta+1, \beta, 1-z\end{matrix}\middle|\, 1\right) = \frac{z(n+2\beta)}{2\beta(z-n)}, \quad n = 0, 1, 2, \ldots, \quad \beta(z-n) \neq 0. \tag{49}$$

$${}_3F_2\left(\begin{matrix}-n, f+1, 1 \\ f, 1-z\end{matrix}\middle|\, \frac{z+f}{n+f}\right) = \frac{z(n+f)}{f(z-n)}, \quad n = 0, 1, 2, \ldots, \quad f(z-n) \neq 0. \tag{50}$$

$${}_3F_2\left(\begin{matrix}f+1, 1, (\lambda+1)f+\lambda z \\ f, 1-z\end{matrix}\middle|\, -\frac{1}{\lambda}\right) = -\frac{\lambda z}{f(\lambda+1)},$$
$$f(\lambda+1) \neq 0, \ |\lambda| > 1 \text{ or } |\lambda| = 1 \text{ and } R(\{(\lambda+1)(f+z)\}) < -1. \tag{51}$$

Equations (50), (51) are the limiting forms of (46) with $e \to \infty$ and $e = z + \lambda n$, $n \to \infty$, respectively. Actually (50) is (51) with $\lambda = -(n+f)/(z+f)$. Note that the limiting form $z \to \infty$ gives nothing new as (46) is symmetric in the parameters e and $1-z$. Also $f \to \infty$ is not recorded as it is a special case of (3). If in (50), we let $n \to \infty$, we get the curious result

$${}_2F_2\left(\begin{matrix}f+1, 1 \\ f, 1-z\end{matrix}\middle|\, -(z+f)\right) = -z/f, \qquad f \neq 0, \tag{52}$$

and with $f = 1$,

$${}_1F_1(2; 1-z; -1-z) = -z \text{ or } {}_1F_1(-1-z; 1-z; 1+z) = -ze^{1+z}. \tag{53}$$

The evaluation of

$${}_{p+1}F_p\left(\begin{matrix}-m, 1+a_p \\ a_p\end{matrix}\middle|\, 1\right)$$

for m a positive integer has already been given, see 2.9(11–15). In a similar fashion if $m \geqslant \sigma$, σ an integer $\geqslant 0$,

$$Q(x) = \frac{\prod_{t=0}^{m-\sigma}(x+\sigma+\mu-1+t)}{(x+\sigma+\mu-1)} \prod_{j=1}^{p}(x+a_j),$$

a polynomial in x of degree $(p + m - \sigma)$, then the mth difference of $Q(x)$ is nil if $m > m + p - \sigma$ or $p < \sigma$, and this leads to the result that if σ and m are integers,

$$\left[\frac{(a_p)}{\Gamma(\mu+\sigma)}\right] {}_{p+2}F_{p+1}\left(\begin{matrix}-m, m+\mu\ 1+a_p\\ \mu+\sigma\ a_p\end{matrix}\middle|\,1\right) = 0, \quad 0 \leqslant p < \sigma \leqslant m \tag{54}$$

Note that 2 9(15) is the special case $\sigma = m$ From 12 2(40, 41) with $t = p$, $\delta = m + \mu$, and $\omega_h = a_h$, we have

$$\left[\frac{(a_p)}{\Gamma(\mu+p+1)}\right] {}_{p+2}F_{p+1}\left(\begin{matrix}-m, m+\mu, 1+a_p\\ p+1+\mu, a_p\end{matrix}\middle|\,1\right)$$
$$= \frac{(-)^p(m+\mu-a_p)\,m!}{(p-m)!\,\Gamma(2m+\mu+1)}\,{}_{p+2}F_{p+1}\left(\begin{matrix}m-p, m+\mu, 1+m+\mu-a_p\\ 2m+\mu+1, m+\mu-a_p\end{matrix}\middle|\,1\right) \tag{55}$$

Note that (54) and (55) are the same when $\sigma = p + 1$ and $m \geqslant p + 1$ In connection with (54) and (55), see 12 2(28–29) See also 12 2(12–15) and 12 4(5–8)

For a Saalschutzian ${}_4F_3$ of unit argument, Bailey (1935) has proved that

$${}_4F_3\left(\begin{matrix}-n\ x, y, z\\ u, v, w\end{matrix}\middle|\,1\right) = \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}\,{}_4F_3\left(\begin{matrix}-n, u-x, u\ y, z\\ u, 1-v+z-n, 1-w+z-n\end{matrix}\middle|\,1\right),$$
$$x + y + z + 1 - n = u + v + w \tag{56}$$

For some other transformation formulas for ${}_{p+1}F_p$'s, $p > 3$, see Bailey (1935) and Slater (1966) See these same sources for the summation of some ${}_{p+1}F_p$'s of argument other than unity For instance,

$${}_4F_3\left(\begin{matrix}a\ 1+\frac{1}{2}a, b, c\\ \frac{1}{2}a\ 1+a-b, 1+a-c\end{matrix}\middle|\,-1\right) = \frac{(1+a-b-c)_b}{(1+a-b)_b} \tag{57}$$

Finally from 2 5(1) and 3 6(1), we have

$${}_2F_1(1, 2a, 2a+1, -1) = a[\psi(a+\tfrac{1}{2}) - \psi(a)] \tag{58}$$

Chapter IV CONFLUENT HYPERGEOMETRIC FUNCTIONS

4.1. Introduction

In Chapter III, we gave some discussion of the ${}_1F_1(a; c; z)$ hypergeometric series, and in 3.5 we showed how this series follows from that of the ${}_2F_1(a, b; c; z/b)$ by the confluence principle. We remarked that this concept is useful to deduce properties of the ${}_1F_1$ from those of the ${}_2F_1$. In particular, see 3.4(28, 29), we used this notion to deduce two recursion formulas for the ${}_1F_1$. In this chapter, we set down other properties and concepts useful for our approximation studies. Our discussion is rather curt, and proofs for the most part are only sketched. For references, see Buchholz (1953), Erdélyi *et al.* (1953, Vol. I, Chapter 6; Vol. II, Chapters 7–9), Luke (1962a), Slater (1960), Tricomi (1954), Watson (1945), and Whittaker and Watson (1927). See also the references noted at the end of 3.2.

4.2. Integral Representations

The analog of Euler's formula [see 3.6(1)] is

$$ {}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1}(1-t)^{c-a-1}\,dt, $$

$$ R(c) > R(a) > 0. \tag{1} $$

This is a special case of 3.6(10) and may be used to prove Kummer's formula 4.4(12). The combination 3.6(13) and 3.1(10) gives

$$ {}_1F_1(\sigma; \nu+1; -\lambda^2/4z^2) = \frac{2\Gamma(\nu+1)\,z^{2\sigma}}{\Gamma(\sigma)(\lambda/2)^\nu} \int_0^\infty e^{-z^2u^2} u^{2\sigma-\nu-1} J_\nu(\lambda u)\,du, $$

$$ R(\sigma) > 0, \qquad R(z^2) > 0. \tag{2} $$

From 3 6(19),

$$\lambda^{c-1}\,{}_1F_1(a, c, \lambda z) = (2\pi i)^{-1}\Gamma(c)\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}t^{-c}(1 - z/t)^{-a}\,dt,$$

$$\lambda \text{ real} \quad \lambda \neq 0, \quad R(c) > 0, \quad \gamma > 0 \quad |\arg(1 - z/\gamma)| < \pi, \tag{3}$$

and the Mellin Barnes integral representation [see 3 6(28)] is

$$\frac{\Gamma(a)}{\Gamma(c)}\,{}_1F_1(a, c, z) = (2\pi i)^{-1}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(a+s)\,\Gamma(-s)(-z)^s}{\Gamma(c+s)}\,ds,$$

$$0 > \gamma > -R(a), \qquad |\arg(-z)| < \pi/2 \tag{4}$$

The formula

$$z^{-1}\{[\delta(\delta + c - 1) - z(\delta + a)][e^{-zt}t^{a-1}(1 + t)^{c-a-1}] = -(\partial/\partial t)[e^{-zt}t^{a}(1 + t)^{c-a}],$$
$$\delta = z\,d/dz \tag{5}$$

shows that

$$\int_C e^{-zt}t^{a-1}(1 + t)^{c-a-1}\,dt \tag{6}$$

is a solution of 4 4(1) if C is either closed on the Riemann surface or terminates at the zeros of $e^{-zt}t^{a}(1 + t)^{c-a}$ Thus we can recover (1) If $R(a) > 0$, $R(z) > 0$, a choice for C is the infinite ray starting from the origin Hence a solution of 4 4(1) is

$$\psi(a, c, z) = [\Gamma(a)]^{-1}\int_0^\infty e^{-zt}t^{a-1}(1 + t)^{c-a-1}\,dt, \quad R(a) > 0, \quad R(z) > 0 \tag{7}$$

We can extend the domain of definition by rotating the path of integration Thus,

$$\psi(a, c, z) = [\Gamma(a)]^{-1}\int_0^{\infty e^{i\theta}} e^{-zt}t^{a-1}(1 + t)^{c-a-1}\,dt,$$

$$R(a) > 0 \quad |\theta| < \pi, \quad |\theta + \arg z| < \pi/2, \quad \text{i e} \quad |\arg z| < 3\pi/2 \tag{8}$$

Now,

$$(1 + t)^{c-a-1} = (2\pi i)^{-1}\int_{\gamma-i\infty}^{\gamma+i\infty}(1 + a - c)_s\,\Gamma(-s)\,t^s\,ds \qquad 0 > \gamma > R(c - a) \tag{9}$$

and if this is combined with (8), and the order of integration is interchanged, which is permissible when $\gamma + R(a) > 0$, we get

$$\psi(a, c, z) = [2\pi i\Gamma(a)]^{-1}\int_{\gamma-i\infty}^{\gamma+i\infty}(1 + a - c)_s\,\Gamma(-s)\int_0^\infty e^{-zt}t^{s+a-1}\,dt\,ds, \tag{10}$$

and so from 2.1(2),

$$\psi(a; c; z) = (2\pi i z^a)^{-1} \int_{\gamma-i\infty}^{\gamma+i\infty} (a)_s (1 + a - c)_s \Gamma(-s) z^{-s} ds, \tag{11}$$

and with a minor change in notation,

$$\psi(a; c; z) = (2\pi i)^{-1} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(a+s)\Gamma(-s)\Gamma(1-c-s) z^s}{\Gamma(a)\Gamma(a-c+1)} ds,$$

$$-R(a) < \gamma < \min(0, 1 - R(c)), \qquad |\arg z| < 3\pi/2. \tag{12}$$

In the derivation, more stringent requirements on the variable and parameters are needed, but these may be relaxed by analytic continuation. Also, the conditions imposed on the parameters may be further relaxed. As in 3.6(28), (4) is valid for any γ if a is not a negative integer or zero, and if the integration contour is indented when necessary so that poles due to $\Gamma(-s)$ lie to the right of the path while those due to $\Gamma(a+s)$ lie to the left of the path. Again, (12) is valid for all γ as long as neither a nor $1 + a - c$ is a negative integer or zero, provided that the path of integration separates the poles of $\Gamma(a+s)$ from those of $\Gamma(-s)\Gamma(1-c-s)$.

4.3. Elementary Relations for the Confluent Functions

The following formulas are either special cases of results in 3.4 or may be deduced from expressions given there by use of the confluence principle.

$$\frac{d^n}{dz^n} {}_1F_1(a; c; z) = \frac{(a)_n}{(c)_n} {}_1F_1\left(\begin{matrix} a+n \\ c+n \end{matrix}\middle| z\right). \tag{1}$$

$$\frac{d^n}{dz^n}[z^\delta {}_1F_1(a; c; z)] = (\delta - n + 1)_n z^{\delta-n} {}_2F_2\left(\begin{matrix} a, \delta+1 \\ c, \delta+1-n \end{matrix}\middle| z\right). \tag{2}$$

$$\frac{d^n}{dz^n}[z^{a+n-1} {}_1F_1(a; c; z)] = (a)_n z^{a-1} {}_1F_1(a+n; c; z). \tag{3}$$

$$\frac{d^n}{dz^n}[z^\delta e^{-z} {}_1F_1(a; c; z)] = (\delta - n + 1)_n z^{\delta-n} {}_2F_2\left(\begin{matrix} c-a, \delta+1 \\ c, \delta+1-n \end{matrix}\middle| -z\right). \tag{4}$$

$$\frac{d^n}{dz^n}\left[z^{n+c-1} e^{-z} {}_1F_1\left(\begin{matrix} a+n \\ c+n \end{matrix}\middle| z\right)\right] = (c)_n z^{c-1} e^{-z} {}_1F_1(a; c; z). \tag{5}$$

$$\frac{d^n}{dz^n}[z^{c-a+n-1} e^{-z} {}_1F_1(a; c; z)] = (c-a)_n z^{c-a-1} e^{-z} {}_1F_1\left(\begin{matrix} a-n \\ c \end{matrix}\middle| z\right). \tag{6}$$

$$\frac{d^n}{dz^n}[e^{-z}\,{}_1F_1(a, c, z)] = \frac{(-)^n(c-a)_n}{(c)_n}\,e^{-z}\,{}_1F_1\left(\begin{matrix} a \\ c+n \end{matrix}\middle| z\right) \tag{7}$$

$$\frac{d^n}{dz^n}[z^\delta e^{-z}] = (\delta - n + 1)_n\, z^{\delta-n}\, e^{-z}\,{}_1F_1\left(\begin{matrix} -n \\ \delta + 1 - n \end{matrix}\middle| z\right) \tag{8}$$

For contiguous relations, we have

$$(c-a)\,{}_1F_1(a-) + (2a - c + z)\,{}_1F_1 - a\,{}_1F_1(a+) = 0, \tag{9}$$

$$-c(c-1)\,{}_1F_1(c-) + c(c-1+z)\,{}_1F_1 - (c-a)z\,{}_1F_1(c+) = 0, \tag{10}$$

$$c(a+z)\,{}_1F_1 - ac\,{}_1F_1(a+) - (c-a)z\,{}_1F_1(c+) = 0, \tag{11}$$

$$c\,{}_1F_1 - c\,{}_1F_1(a-) - z\,{}_1F_1(c+) = 0, \tag{12}$$

$$(c-a-1)\,{}_1F_1 + a\,{}_1F_1(a+) - (c-1)\,{}_1F_1(c-) = 0, \tag{13}$$

$$(a-1+z)\,{}_1F_1 + (c-a)\,{}_1F_1(a-) - (c-1)\,{}_1F_1(c-) = 0 \tag{14}$$

Clearly, if m and n are integers, ${}_1F_1(a+m, c+n, z)$ can be expressed as a linear combination of ${}_1F_1$ and one of its contiguous functions with coefficients which are rational functions of a, c, and z Note that (1)–(8), or the latter in combination with (9)–(14), give rise to differential-difference properties

The following formulas for the ψ function may be proved from appropriate integral representations in 4 2 or from the representation 4 5(2) and results in (1)–(14)

$$\frac{d^n}{dz^n}\psi(a, c, z) = (-)^n(a)_n\,\psi(a+n, c+n, z) \tag{15}$$

$$\frac{d^n}{dz^n}[z^c\,{}^1\psi(a, c, z)] = (-)^n(1+a-c)_n\, z^{c-n}\,{}^1\psi(a, c-n, z) \tag{16}$$

$$\frac{d^n}{dz^n}[z^{a+n-1}\psi(a, c, z)] = (a)_n(1+a-c)_n\, z^{a-1}\psi(a+n, c, z) \tag{17}$$

$$\frac{d^n}{dz^n}[e^{-z}\psi(a, c, z)] = (-)^n\, e^{-z}\psi(a, c+n, z), \tag{18}$$

$$\frac{d^n}{dz^n}[e^{-z}z^{c-a+n-1}\psi(a, c, z)] = (-)^n\, e^{-z}z^{c-a}\,{}^1\psi(a-n, c, z) \tag{19}$$

$$\frac{d^n}{dz^n}[e^{-z}z^c\,{}^1\psi(a, c, z)] = (-)^n\, e^{-z}z^{c-n-1}(1-a)_n\,\psi(a-n, c-n, z) \tag{20}$$

$$\psi(a-) - (2a - c + z)\psi + a(a-c+1)\,\psi(a+) = 0 \tag{21}$$

$$(c-a-1)\,\psi(c-)-(c-1+z)\psi+z\psi(c+)=0. \tag{22}$$

$$(a+z)\psi+a(c-a-1)\,\psi(a+)-z\psi(c+)=0. \tag{23}$$

$$(c-a)\psi+\psi(a-)-z\psi(c+)=0. \tag{24}$$

$$\psi-a\psi(a+)-\psi(c-)=0. \tag{25}$$

$$(a-1+z)\psi-\psi(a-)+(a-c+1)\,\psi(c-)=0. \tag{26}$$

Here $\psi(a\pm)$ stands for $\psi(a\pm 1;\,c;\,z)$, etc.

4.4. Confluent Differential Equation

The linearly independent solutions of

$$[zD^2+(c-z)\,D-a]\,w(z)=0, \qquad D=d/dz, \tag{1}$$

or

$$[\delta(\delta+c-1)-z(\delta+a)]\,w(z)=0, \qquad \delta=zD, \tag{2}$$

are proportional to

$$w_1={}_1F_1\left(\begin{matrix}a\\c\end{matrix}\middle|\,z\right) \quad\text{and}\quad w_2=z^{1-c}\,{}_1F_1\left(\begin{matrix}1+a-c\\2-c\end{matrix}\middle|\,z\right), \tag{3}$$

provided c is not an integer or zero. Sometimes the notation

$$\Phi(a;\,c;\,z)={}_1F_1(a;\,c;\,z) \tag{4}$$

is used (see also 4.9). It follows from (1) that the differential equation

$$hy''+\left\{\frac{2\alpha h}{z}+2f'h-\frac{hh''}{h'}-hh'+ch'\right\}y'$$
$$+\left\{h'\left(\frac{\alpha}{z}+f'\right)(c-h)+h\left[\frac{\alpha(\alpha-1)}{z^2}+\frac{2\alpha f'}{z}+f''+(f')^2\right.\right.$$
$$\left.\left.-\frac{h''}{h'}\left(\frac{\alpha}{z}+f'\right)\right]-a(h')^2\right\}y=0$$

is satisfied by

$$y=z^{-\alpha}\,e^{-f(z)}\,{}_1F_1\left(\begin{matrix}a\\c\end{matrix}\middle|\,h(z)\right) \tag{5}$$

where $f=f(z)$, etc.

As remarked in 3 1(10), the Bessel function of the first kind $J_\nu(z)$ can be expressed in terms of a ${}_0F_1$, and the latter may be viewed as a confluent form of the ${}_1F_1$ The function $J_\nu(z)$ can also be expressed in terms of a ${}_1F_1$ We have the two representations

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(\,;\nu+1,\,-z^2/4) \qquad I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(\,;\nu+1,\,z^2/4), \tag{6}$$

$$\begin{aligned} J_\nu(z) &= \frac{(z/2)^\nu\, e^{\pm iz}}{\Gamma(\nu+1)}\,{}_1F_1(\tfrac{1}{2}+\nu,\,1+2\nu,\,\mp 2iz) \\ I_\nu(z) &= \frac{(z/2)^\nu\, e^{\pm z}}{\Gamma(\nu+1)}\,{}_1F_1(\tfrac{1}{2}+\nu,\,1+2\nu,\,\mp 2z) \end{aligned} \tag{7}$$

Other special cases of confluent hypergeometric functions and appropriate references are noted in 4 9, and properties of these and Bessel functions are listed in Chapter VI Here we remark that the differential equation

$$\begin{aligned} ky'' &+ \left\{\frac{2\beta k}{z} + 2kg' - \frac{k''k}{k} + k'\right\} y' \\ &+ \left\{k\left[\frac{\beta(\beta-1)}{z^2} + \frac{2\beta g}{z} + g'' + (g')^2 - \frac{k'}{k}\left(\frac{\beta}{z} + g'\right)\right]\right. \\ &\qquad \left. + k'\left(\frac{\beta}{z} + g'\right) + k(k')^2\left(1 - \frac{\nu^2}{k^2}\right)\right\} y = 0 \end{aligned} \tag{8}$$

is satisfied by

$$y = z^{-\beta} e^{-g(z)} C_\nu(k), \tag{9}$$

where $y = y(z)$, etc , and $C_\nu(z) = AJ_\nu(z) + BY_\nu(z)$, A and B are independent of z, and $Y_\nu(z)$ is the Bessel function of the second kind

We return to the solutions of (1) If $a = -m$, m a positive integer or zero, and c is a negative integer such that $m + c < 1$, then a well defined solution of (1) is

$$\begin{aligned} f(a, c, z) &= \sum_{k=0}^{-a} \frac{(a)_k z^k}{(c)_k k!} = \sum_{k=0}^{m} \frac{(-m)_k z^k}{(1-s)_k k!} \\ &= \frac{(s-1-m)!\, z^m}{(s-1)!}\,{}_2F_0(-m,\,s-m,\,-z^{-1}) \end{aligned}$$

$$c = 1 - s, \qquad s \text{ a positive integer}, \qquad m < s \tag{10}$$

and this is independent of w_2. Again if a and c are positive integers, $a < c$, then

$$z^{1-c}f(1+a-c;2-c;z) = z^{-s}\sum_{k=0}^{s-m}\frac{(m-s)_k z^k}{(1-s)_k k!}$$

$$= \frac{(m-1)!z^{-m}}{(s-1)!}\,{}_2F_0(m-s,m;-z^{-1}),$$

$$a = m, \qquad m \text{ a positive integer,}$$

$$c = 1+s, \qquad s \text{ a positive integer or zero,} \qquad m < s+1, \tag{11}$$

is a well-defined solution of (1) and is independent of w_2. Observe that (10), (11) are confluent forms of 3.7(14, 15), respectively. A thorough study of the complete solution of (1) is deferred to 4.5.

From 3.8(2) and the confluence principle, or directly from 4.2(1),

$$w_1 = {}_1F_1(a;c;z) = e^z{}_1F_1(c-a;c;-z), \tag{12}$$

$$w_2 = z^{1-c}\,{}_1F_1(1+a-c;2-c;z) = z^{1-c}e^z\,{}_1F_1(1-a;2-c;-z). \tag{13}$$

This is known as Kummer's transformation.

4.5. The Complete Solution

If c is not an integer, then 4.4(12, 13) are independent solutions of 4.4(1). The function $w_3 = \psi(a;c;z)$ is also a solution, and from 4.2(12), we see that

$$w_3 = \psi(a;c;z) = z^{1-c}\psi(1+a-c;2-c;z). \tag{1}$$

These three solutions cannot be independent. If c is not an integer, the poles of the integrand in 4.2(12) are simple, and evaluation of this integral as a sum of residues at these poles gives

$$\begin{aligned} w_3 &= \psi(a;c;z) = \frac{\Gamma(1-c)}{\Gamma(1+a-c)}w_1 + \frac{\Gamma(c-1)}{\Gamma(a)}w_2, \\ w_1 &= {}_1F_1\left(\begin{matrix}a\\c\end{matrix}\middle|\, z\right), \qquad w_2 = z^{1-c}\,{}_1F_1\left(\begin{matrix}1+a-c\\2-c\end{matrix}\middle|\, z\right). \end{aligned} \tag{2}$$

If we replace a by $c-a$ and z by $ze^{-i\epsilon\pi}$, $\epsilon = \pm 1$, and use 4.4(12, 13), then

$$w_4 = e^z\psi(c-a;c;ze^{-i\epsilon\pi}) = \frac{\Gamma(1-c)}{\Gamma(1-a)}w_1 - \frac{\Gamma(c-1)}{\Gamma(c-a)}e^{i\epsilon\pi c}w_2, \tag{3}$$

and by elimination,

$$w_1 = \frac{\Gamma(c)}{\Gamma(c-a)} e^{i\epsilon\pi a} w_3 + \frac{\Gamma(c)}{\Gamma(a)} e^{i\epsilon\pi(a-c)} w_4, \tag{4}$$

$$w_2 = e^{i\epsilon\pi(a-c)} \left\{ -\frac{\Gamma(2-c)}{\Gamma(1-a)} w_3 + \frac{\Gamma(2-c)}{\Gamma(1+a-c)} w_4 \right\} \tag{5}$$

In (4), (5), the convention

$$\begin{aligned} \epsilon = \operatorname{sign}(I(z)) &= 1 \qquad \text{if } I(z) > 0, \\ &= -1 \qquad \text{if } I(z) < 0, \end{aligned} \tag{6}$$

is often used

Thus any two of the four quantities w_1, w_2, w_3, and w_4 form a fundamental system if a, c, and $c - a$ are not integers If a is a negative integer or zero, w_3 and w_4 differ only by a constant multiple The same is true for w_3 and w_2 if $c - a$ is a positive integer Again if a is a positive integer w_4 is a constant multiple of w_2, and likewise for w_4 and w_1 if $c - a$ is a negative integer or zero If c is an integer, zero included, either $w_1 = w_2$ or one of these is not defined If c is a negative integer or zero, say $c = 1 - n$, $n = 1, 2, 3, \ldots$, then

$$\lim_{c \to 1-n} \frac{w_1}{\Gamma(c)} = \frac{(a)_n}{n!} z^n {}_1F_1(a+n, 1+n, z) \tag{7}$$

satisfies 4 4(1), but this is a multiple of w_2, and so we get no new solution The derivation of (2) assumes that c is not an integer, but clearly this is not essential, for by continuity it holds also for integer c Indeed, as in the ${}_2F_1$ studies, we can use (2) to derive a logarithmic solution of 4 4(1) when c is an integer For the present situation, we can use the developments for $w_3 = H(z)$ [see 3 10(3, 4, 20)] provided we treat all terms involving a_2 as empty That is, if a term involving a_2 appears as a product, treat it as unity, and if it appears as a sum, treat it as zero We find

$$\begin{aligned} w_3 = \psi(a, 1+s, z) = \frac{z^{-s}}{\Gamma(a)} \sum_{k=0}^{s-1} \frac{(a-s)_k}{k!} (s-1-k)!(-)^k z^k \\ + \frac{(-)^{s+1}}{s!\Gamma(a-s)} \Big\{ [\gamma + \ln z + \psi(a) - \psi(1+s)]\, {}_1F_1\left(\begin{matrix} a \\ s+1 \end{matrix} \middle| z\right) \\ + \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(1+s)_k k!} (\psi(a+k) - \psi(a) - \psi(1+s+k) + \psi(1+s) \\ - \psi(1+k) + \psi(1)) \Big\} \end{aligned} \tag{8}$$

where a is not a negative integer or zero and s is a positive integer or zero

An alternative proof of (8) follows from 4.2(12) in the same manner as 3.10(13) follows from 3.6(28); see 3.10(14, 15).

If a is a positive integer and $a \leqslant s$, then only the polynomial part of (8) remains. In this event, $w_3 = \psi(a; 1 + s; z)$ is a multiple of w_2 truncated after s terms as may be deduced from (2), or with an apparent change of notation, w_3 is proportional to 4.4(11). Thus the restriction on a is not essential. The point is that we have no logarithmic solution if a is not so restricted.

The logarithmic solution for $\psi(a; 1 - s; z)$ follows readily from (8) in view of (1).

In a similar fashion from the developments for $w_6 = V(z)$ [see 3.10(1, 2, 13)] we get

$$\begin{aligned} w_4 &= e^z\psi(1 + s - a; 1 + s; ze^{-i\epsilon\pi}) \\ &= \frac{(-)^s z^{-s}}{\Gamma(1 + s - a)} \sum_{k=0}^{s-1} \frac{(a - s)_k (s - 1 - k)!(-)^k z^k}{k!} \\ &\quad + \frac{(-)^{s+1}}{s!\Gamma(1 - a)} \Big\{[\gamma + \ln(ze^{-i\epsilon\pi}) + \psi(1 - a) - \psi(1 + s)]\, {}_1F_1\left(\begin{matrix} a \\ 1 + s \end{matrix}\middle| z\right) \\ &\quad + \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(1 + s)_k k!} (\psi(a + k) - \psi(a) - \psi(1 + s + k) + \psi(1 + s) \\ &\qquad\qquad - \psi(1 + k) + \psi(1))\Big\}, \end{aligned} \tag{9}$$

where a is not an integer or zero and s is a positive integer or zero.

If a is a positive integer, then w_4 is a linear combination of both w_1 and w_2 unless $a > s$, in which case w_4 is a multiple of w_1. This follows from (2) and may also be deduced from (9). If a is a negative integer or zero, then from (9) and 2.4(13, 14) or from 3.10(18), we have

$$\begin{aligned} e^z\psi(1 + s + m; 1 + s; ze^{-i\epsilon\pi}) &= \frac{(-)^s z^{-s}}{(m + s)!} \sum_{k=0}^{s-1} \frac{(-m - s)_k (s - 1 - k)!(-)^k z^k}{k!} \\ &\quad + \frac{(-)^{s+1}}{s!m!} \Big\{[\gamma + \ln(ze^{-i\epsilon\pi}) + \psi(1 + m) \\ &\quad - \psi(1 + s)]\, {}_1F_1\left(\begin{matrix} -m \\ 1 + s \end{matrix}\middle| z\right) \\ &\quad + \sum_{k=0}^{m} \frac{(-m)_k z^k}{(1 + s)_k k!} (\psi(1 + m - k) - \psi(1 + m) \\ &\quad - \psi(1 + s + k) + \psi(1 + s) - \psi(1 + k) + \psi(1)) \\ &\quad + (-)^m m! \sum_{k=m+1}^{\infty} \frac{(k - m - 1)!\, z^k}{(1 + s)_k k!}\Big\}, \end{aligned} \tag{10}$$

where m and s are positive integers or zero.

Similar expansions for $e^z\psi(1-s-a, 1-s, ze^{-i\epsilon\pi})$ can be obtained from (9) and (10) in view of (1)

Let W_{ij} denote the Wronskian of the solutions w_i and w_j Thus,

$$W_{ij} = W(w_i, w_j) = w_i(z)\, w_j(z) - w_i(z)\, w_j(z) \tag{11}$$

Also let

$$B = z^{-c}e^z \tag{12}$$

Then,

$$W_{12} = (1-c)\,B, \qquad W_{13} = -\frac{\Gamma(c)}{\Gamma(a)}B, \tag{13}$$

$$W_{14} = \frac{\Gamma(c)}{\Gamma(a)}e^{i\pi c}B \qquad W_{23} = -\frac{\Gamma(c)}{\Gamma(1+a-c)}B, \tag{14}$$

$$W_{24} = -\frac{\Gamma(c)}{\Gamma(1-a)}B \qquad W_{34} = \frac{B}{\sin \pi c}[e^{i\pi c}\sin \pi(c-a) + \sin \pi a] \tag{15}$$

ϵ as in (6)

4 6 Kummer-Type Relations for the Logarithmic Solutions

4 6 1 Introduction

The Kummer relations for the ${}_1F_1$ and the ψ functions which are solutions to 4 4(1) are given by 4 4(12) and 4 5(1), respectively Here the corresponding relations are developed for the logarithmic solutions to 4 4(1) Again the analysis is due to Norlund, and closely parallels the developments for the ${}_2F_1$ case (see 3 11)

4 6 2 The Case Where c Is a Positive Integer

Throughout this section we assume that

$$c = s+1, \quad s \text{ a positive integer or zero}, \quad a \neq 1\ 2 \quad , s, \tag{1}$$

unless stated to the contrary Let

$$G_1(a, c, z) = (-)^{s+1}z^{-s}(a)_{-s}\,{}^{1}\sum_{k=0}^{s\ 1}\frac{(a-s)_k(s-1-k)!(-)^k z^k}{k!}$$

$$= \frac{s}{z(a-1)}\,{}_3F_1\left(\begin{matrix}1-s\ 1, 1\\ 2-a\end{matrix}\middle|\ -z\ {}^{1}\right) \tag{2}$$

$$G(a; c; z) = G_1(a; c; z) + \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(s+1)_k k!}$$
$$\times \{\ln z + \psi(a+k) - \psi(a) + \psi(1+s) - \psi(1+s+k)$$
$$- \psi(1+k) + \psi(1)\}, \tag{3}$$

$$g(a; c; z) = G_1(a; c; z) + \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(s+1)_k k!}$$
$$\times \{\ln z + \psi(a+k) - \psi(1+s+k) - \psi(1+k)\}. \tag{4}$$

In (3), (4), $|\arg z| < \pi$.

$$g_1(a; c; z) = G_1(a; c; z) + \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(s+1)_k k!}$$
$$\times \{\ln(ze^{-i\epsilon\pi}) + \psi(1-a-k) - \psi(1+s+k) - \psi(1+k)\},$$
$$\epsilon = \pm 1, \qquad -(1-\epsilon)\pi < \arg z < (1+\epsilon)\pi. \tag{5}$$

Then,

$$g(a; c; z) = G(a; c; z) + [\psi(a) - \psi(c) - \psi(1)]\, {}_1F_1(a; c; z),$$
$$|\arg z| < \pi, \tag{6}$$

$$g_1(a; c; z) = G(a; c; z) + [-\delta i\pi + \psi(1-a) - \psi(c) - \psi(1)]\, {}_1F_1(a; c; z), \tag{7}$$

$$g_1(a; c; z) - g(a; c; z) = \frac{\pi e^{-i\delta\pi a}}{\sin \pi a}\, {}_1F_1(a; c; z),$$
$$\delta = \pm 1, \qquad -(1-\delta)\,\pi/2 < \arg z < (1+\delta)\,\pi/2, \tag{8}$$

and from 4.5(8, 9),

$$g(a; c; z) = (-)^{s+1} s!\, \Gamma(a-s)\, \psi(a; 1+s; z), \tag{9}$$

$$g_1(a; c; z) = (-)^{s+1} s!\, \Gamma(1-a)\, e^z \psi(1+s-a; 1+s; ze^{-i\epsilon\pi}). \tag{10}$$

In (4), (6), and (9), a is not a negative integer or zero; in (5), (7), and (10), a is not a positive integer; and in (8), a is not an integer. Clearly each of the equations (3)–(10) satisfies 4.4(1).

Relations analogous to 4.3(1) and to 4.3(4) (there put $\delta = c - 1$) are

$$\frac{d^n}{dz^n} g(a; c; z) = \frac{(a)_n}{(c)_n} g(a+n; c+n; z), \tag{11}$$

$$\frac{d^n}{dz^n} \{z^{c-1} g(a; c; z)\} = (-)^n (1-c)_n z^{c-n-1} g(a; c-n; z), \tag{12}$$

as may be verified by differentiation In (11), (12), g may be replaced by g_1 or G

Since $e^z G(c-a, c, -z)$ is also a solution of 4 4(1), then after the manner of proving 3 11 2(22), or by confluence, we find with δ as in (8),

$$G(a, c, z) = e^z G(c-a, c, -z) + [\delta i\pi + \psi(1-a+s) - \psi(1-a)]\, {}_1F_1(a, c, z) \tag{13}$$

Also,

$$g(a, c, z) = e^z g_1(c-a, c, -z), \qquad a \text{ not an integer} \leqslant s, \tag{14}$$

$$g_1(a, c, z) = e^z g(c-a, c, -z), \qquad a \text{ not a positive integer}, \tag{15}$$

$$g(a, c, z) = e^z g(c-a, c, -z) - \frac{\pi e^{-\delta i\pi a}}{\sin \pi a}\, {}_1F_1(a, c; z), \tag{16}$$

$$g_1(a, c, z) = e^z g_1(c-a, c, -z) + \frac{\pi e^{-\delta i\pi a}}{\sin \pi a}\, {}_1F_1(a, c, z), \tag{17}$$

where in (16), (17) a is not an integer and δ has the same meaning as in (8)

Both $g(a, c, z)$ and ${}_1F_1(a, c, z)$ satisfy the same contiguous relations 4 3(9–14)

4 6 3 The Case Where c Is a Negative Integer or Zero

If c is not an integer, w_1 and w_2 [see 4 4(3)] are independent solutions of 4 4(1) If c is a negative integer or zero, and a does not coalesce with the numbers $0, -1, \ldots, c$, then the Kummer-type relations follow from those of the previous section as the roles of w_1 and w_2 are essentially interchanged Indeed, with

$$c = 1 - s, \qquad s \text{ a positive integer or zero}, \tag{1}$$

$$z^{1-c} g(1+a-c, 2-c, z) = (-)^{s+1} s!\, \Gamma(a)\, \psi(a, 1+s, z), \tag{2}$$

where a is not a negative integer or zero, and

$$z^{1-c} g_1(1+a-c, 2-c, z) = -s!\, \Gamma(1-s-a)\, e^z \psi(1-s-a, 1-s, ze^{-i\pi}), \tag{3}$$

provided $(a+s)$ is not a positive integer

Let us now suppose that $c = 1 - s$ is a negative integer and that $a = -m$, m a positive integer or zero with $m < s$ Then using the notation of 4 4(10), we have

$$f(a, c, z) = e^z f(c-a, c, -z)$$
$$+ \frac{(-)^{m+s}(s-m-1)!\, m!\, z^{1-c}}{s!\,(s-1)!}\, {}_1F_1(1+a-c, 2-c, z) \tag{4}$$

4.7. Asymptotic Expansions for Large z

The representation 4.2(7) satisfies the conditions of 1.4(2) and so upon expanding $(1+t)^{c-a-1}$ by the binomial series and termwise integrating, we find that

$$\psi(a;c;z) \sim z^{-a}\,{}_2F_0(a, 1+a-c; -z^{-1}),$$
$$|z| \to \infty, \qquad |\arg z| \leqslant 3\pi/2 - \epsilon, \qquad \epsilon > 0. \tag{1}$$

The ${}_2F_0$ is divergent but has significance in that it is the asymptotic representation of $z^a\psi(a;c;z)$. Later we shall see that every nonterminating ${}_pF_q$, $p > q+1$, is the asymptotic expansion of a well-defined function in some sector of the complex plane. To get an estimate of the error when the asymptotic expansion is truncated after n terms, we notice that

$$(1+t)^{c-a-1} = \sum_{k=0}^{n-1} \frac{(1+a-c)_k(-)^k t^k}{k!} + S_n(t),$$

$$S_n(t) = \frac{(1+a-c)_n(-)^n t^n (1+t)^{-1}}{n!}\,{}_2F_1\left(\begin{matrix} c-a, 1 \\ n+1 \end{matrix}\,\middle|\, \frac{t}{t+1}\right).$$

Thus,

$$\psi(a;c;z) = z^{-a}\sum_{k=0}^{n-1} \frac{(a)_k(1+a-c)_k(-)^k z^{-k}}{k!} + R_n(z),$$
$$R_n(z) = \frac{(-)^n(1+a-c)_n}{\Gamma(a)\,n!}\int_0^\infty \frac{e^{-zt}t^{a+n-1}}{(1+t)}\,{}_2F_1\left(\begin{matrix} c-a, 1 \\ n+1 \end{matrix}\,\middle|\, \frac{t}{t+1}\right) dt. \tag{2}$$

Clearly $R_n(z) = O(|z^{-a-n}|)$, $|z| \to \infty$, $|\arg z| \leqslant \pi/2 - \epsilon$, $\epsilon > 0$, and by rotation of the path of integration, we get the statement (1). Now in the integrand of (2),

$${}_2F_1\left(\begin{matrix} c-a, 1 \\ n+1 \end{matrix}\,\middle|\, \frac{t}{t+1}\right) = 1 + O(n^{-1})$$

and so,

$$|R_n(z)| \leqslant \left|\frac{(1+a-c)_n(a)_n}{n!}\right| x^{-a_1-n}, \qquad x = R(z) > 0, \quad a_1 = R(a). \tag{3}$$

Thus if a, c, and z are real, the error does not exceed the first term neglected and is of the same sign as the first term neglected.

The asymptotic expansion for ${}_1F_1(a\ c\ z)$ follows from 4 5(4)

$$
{}_1F_1(a\ c\ z) \sim \frac{\Gamma(c)}{\Gamma(c-a)}(z^{-1}e^{i\ \pi})^a\,{}_2F_0(a\ 1+a-c\ -z^{-1})
$$

$$
+\frac{\Gamma(c)}{\Gamma(a)}e^z z^{a\ c}\,{}_2F_0(c-a\ 1-a\ z^{-1})
$$

$$
\epsilon = 1 \quad \text{if} \quad I(z) > 0 \qquad \epsilon = -1 \quad \text{if} \quad I(z) < 0 \qquad |z| \to \infty \qquad |\arg z| < \pi \tag{4}
$$

$$
{}_1F(a\ c\ z) \sim \frac{\Gamma(c)}{\Gamma(a)}e^z z^{a\ c} \qquad \text{as} \quad R(z) \to \infty \tag{5}
$$

$$
{}_1F_1(a\ c\ z) \sim \frac{\Gamma(c)}{\Gamma(c-a)}(\ \ z)^{-a} \quad \text{as} \quad R(z) \to -\infty \tag{6}
$$

We can also write

$$
{}_1F_1(a\ c\ z) \sim \frac{\Gamma(c)}{\Gamma(a)}e^z z^{a\ c}\,{}_2F_0(c-a\ 1-a\ z^{-1})
$$

$$
+\frac{\Gamma(c)}{\Gamma(c-a)}z^{-a}e^{\ i\delta a\pi}\,{}_2F_0(a\ 1+a-c\ -z^{-1})
$$

$$
|z| \to \infty \qquad \delta = \pm 1 \qquad -(2+\delta)\pi/2 < \arg z < (2-\delta)\pi/2 \tag{7}
$$

If $|\arg z| < \pi/2$ the apparent discrepancy in (7) is a case of Stoke s phenomena See Erdelyi (1956) Heading (1962) Jeffreys (1962) and Watson (1945)

From 4 6 2(9) and (1) we have

$$
g(a\ c\ z) \sim (-)^{s+1}s!\Gamma(a-s)\,z^{\ a}\,{}_2F_0(a\ a-s\ -z^{-1}) \tag{8}
$$

where $|z| \to \infty$ $|\arg z| < 3\pi/2$ a and s are as in 4 6 2(9) Similarly 4 6 2(10) and (1) yield

$$
g(a\ c\ z) \sim \frac{e^{-\ \pi a}s!\Gamma(1-a)\,e^z}{z^{1+s\ a}}\,{}_2F_0(1+s-a\ 1-a\ z^{-1}) \tag{9}
$$

where $|z| \to \infty$ $\epsilon = \pm 1$ $|\arg(ze^{-i\ \pi})| < 3\pi/2$ a and s are as in 4 6 2(10) The combination 4 6 2(6) (7) and (8) gives

$$
G(a\ c\ z) \sim \frac{s![\psi(1+s)+\psi(1)-\psi(a)]\,e^z}{\Gamma(a)\,z^{1\ s\ a}}\,{}_2F_0(1+s\ \ a\ 1-a\ z^{-1})
$$

$$
+\frac{(\ \ )^{s+1}s!\Gamma(a-s)}{(ze^{i\delta\pi})^a}
$$

$$
\times\left\{e^{\ \delta\pi a}\ [\psi(1+s)+\psi(1)\ \ \psi(a)]\frac{\sin\pi a}{\pi}\right\}{}_2F_0(a\ a-s\ \ z^{-1})
$$

$$
|z| \to \infty \qquad \delta = \pm 1 \qquad -(2+\delta)\pi/2 < \arg z < (2-\delta)\pi/2 \tag{10}
$$

4.8. Asymptotic Behavior for Large Parameters and Variable

From 4.4(12),

$$ {}_1F_1(a; c; z) = 1 + O(c^{-1}), \qquad a, z \text{ bounded}, \qquad c \to \infty, \tag{1} $$

$$ {}_1F_1(a; c; z) = e^z[1 + O(c^{-1})], \qquad c - a, z \text{ bounded}, \quad c \to \infty. \tag{2} $$

The behavior of $\psi(a; c; z)$ in these cases follows from 4.5(1, 2) and 2.11(2, 11). It is sufficient to record

$$ \begin{aligned} \psi(a; c; z) &= (-c)^{-a}[1 + O(c^{-1})] \\ &\quad + \frac{(2\pi)^{1/2} z^{1-c}}{\Gamma(a)} \exp[z - c + (c - 3/2) \ln c][1 + O(c^{-1})], \\ a, z \text{ bounded}, &\qquad c \to \infty, \qquad |\arg \pm c| \leqslant \pi - \epsilon, \qquad \epsilon > 0. \end{aligned} \tag{3} $$

The expansion 3.5(46) affords another representation. We have

$$ \begin{aligned} {}_1F_1(a; c; z) &= v^a \Big[1 - \frac{a(a+1)}{2c}(uv)^2 + \frac{a(a+1)}{24c^2}(uv)^2 \{12 + 16(a+2)uv \\ &\quad + 3(a+2)(a+3)(uv)^2\}\Big] + O(c^{-3}), \\ u = z/c, \quad v = (1-u)^{-1}; &\qquad |z| < |c| \quad \text{if} \quad R(c) \geqslant 0; \\ |z| < |c| \sin\delta, \quad 0 < \delta \leqslant \pi/2 &\quad \text{if} \quad R(c) \leqslant 0. \end{aligned} \tag{4} $$

The case when $a \to \infty$ is quite complicated. Here we can deduce an interesting representation from 3.5(21). There put $p = 0, q = 1$, $\rho_1 = c, \sigma = a$, and replace z by az. Then in terms of the modified Bessel function [see 4.4(6)] we have

$$ \begin{aligned} {}_1F_1(a; c; z) &= \Gamma(c)(az)^{-(c-1)/2}[I_{c-1}(2(az)^{1/2}) + (z/2) I_{c+1}(2(az)^{1/2}) + \cdots], \\ &\qquad |az| < \infty. \end{aligned} \tag{5} $$

This is a special case of a result due to Tricomi (1954). Let

$$ E_\nu(z) = z^{-\nu/2} J_\nu(2z^{1/2}). \tag{6} $$

Then

$$ \frac{e^{-hz}}{\Gamma(c)} {}_1F_1(a; c; z) = \sum_{m=0}^{\infty} C_m z^m E_{m+c-1}(\omega z), \qquad |z| < \infty, \tag{7} $$

$$ {}_2F_1\left(\begin{matrix} a & b \\ c & \end{matrix}\middle| \frac{z}{hz+\omega}\right) = e^{-z}(1+hz/\omega)^b \sum_{m=0}^{\infty} \frac{(b)_m}{(c)_m} C_m (z/\omega)^m $$

$$ \times {}_1F_1\left(\begin{matrix} c-b \\ m+c \end{matrix}\middle| z\right), $$

$$ |z| < |\omega| r, \qquad r = \min(|h^{-1}|, |(1-h)^{-1}|), \tag{8} $$

$$ (1+hz)^a\, {}^c[1+(h-1)z]^{-a} e^{\omega z} = \sum_{m=0}^{\infty} C_m z^m, \qquad |z| < r, \tag{9} $$

where C_m depends only on the parameters a, c, h, and ω By direct computation

$$ C_0 = 1, \qquad C_1 = \omega + a - ch, \qquad 2C_2 = C_1^2 + ch^2 - 2ah + a, \tag{10} $$

and we have the recurrence formula

$$ \begin{aligned} (m+1)\, C_{m+1} = {} & [C_1 - m(2h-1)]\, C_m \\ & + [\omega(2h-1) - h(h-1)(m+c-1)]\, C_{m-1} \\ & + \omega h(h-1)\, C_{m-2} \end{aligned} \tag{11} $$

Here ω and h are free parameters This freedom permits the development of expansions useful for the evaluation of the confluent function for large values of the parameters a and c The following cases are of interest

CASE I $h = 1/2,\ \omega = k = c/2 - a$

$$ \frac{e^{-z/2}}{\Gamma(c)}\, {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\middle| z\right) = \sum_{m=0}^{\infty} B_m (z/2)^m E_{c+m-1}(kz), \tag{12} $$

$$ B_0 = 1, \qquad B_1 = 0, \qquad B_2 = c/2, \qquad B_3 = -2k/3, \qquad B_4 = c(c+2)/8, $$

$$ B_5 = -\frac{k(5c+6)}{15}, \qquad B_6 = \frac{c(c+2)(c+4)}{48} + \frac{2k^2}{9}, \quad , $$

$$ (m+1)\, B_{m+1} = (m+c-1)\, B_{m-1} - 2kB_{m-2} \tag{13} $$

CASE II $\omega = -a$

$$ \frac{e^{-hz}}{\Gamma(c)}\, {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\middle| z\right) = \sum_{m=0}^{\infty} A_m z^m E_{m+c-1}(-az), \tag{14} $$

$$ A_0 = 1, \qquad A_1 = -hc, \qquad A_2 = h^2 c(c+1)/2 + a(\tfrac{1}{2} - h) $$

$$ \begin{aligned} (m+1)\, A_{m+1} = {} & [m(1-2h) - ch]\, A_m \\ & + [a(1-2h) - h(h-1)(m+c-1)]\, A_{m-1} \\ & - ah(h-1)\, A_{m-2}\,. \end{aligned} \tag{15} $$

Note that (13) and (15) reduce to three-term recursion formulas if $h = 0$ or 1. If $h = 0$, (14) and (5) are the same.

We first establish (7)–(11) formally and then prove convergence under the conditions stated. If we expand both sides of (7) in powers of z, it is easy to see that the coefficients C_0, C_1, and C_2 are as noted in (10). Multiply both sides of (7) by $z^{b-1}e^{(h-s)z}$ and integrate with respect to z from 0 to ∞. Then using 3.6(13) and 4.4(12), we arrive at (8) when we replace s by $(hz + \omega)/z$. The conditions for validity of the transforms and for expansion of the ${}_2F_1$ on the left of (8) in powers of its argument are

$$R(b) > 0, \qquad |hz + \omega| > |z|, \qquad R(z/\omega) > 0.$$

Next we notice from (7) that the coefficients C_m are independent of b. Put $b = c$ in (8). Then with z replaced by ωz, we get (9).

To see that the series on the right of (7) converges for all z, we observe that the radius of convergence of the two series

$$\sum_{m=0}^{\infty} a_m z^m E_{\alpha+m}(z) \qquad \text{and} \qquad \sum_{m=0}^{\infty} a_m z^m/m!$$

is the same. For since

$$E_\nu(z) = \frac{1}{\Gamma(\nu+1)}[1 + O(|\nu^{-1}|)], \qquad \frac{m!}{\Gamma(m+\alpha)} = m^{1-\alpha}[1 + O(m^{-1})],$$

we have

$$|a_m E_{\alpha+m}(z)|^{1/m} = \left|\frac{a_m}{m!}\right|^{1/m} e^{(1-\alpha)(\ln m)/m}[1 + O(m^{-1})]^{1/m},$$

and so,

$$\overline{\lim_{m\to\infty}}\, |a_m E_{\alpha+m}(z)|^{1/m} = \overline{\lim_{m\to\infty}}\, |a_m/m!\,|^{1/m}.$$

Now the series in (9) has nonzero radius of convergence, and so the radius of convergence of a series whose general term is $C_m z^m/m!$ is infinite and so (7) converges for all z.

For the convergence of the series on the right of (8), we can prove as above that the two series

$$\sum_{m=0}^{\infty} a_m z^m\, {}_1F_1(a; c+m; z) \qquad \text{and} \qquad \sum_{m=0}^{\infty} a_m z^m$$

have the same radius of convergence. Thus the series on the right of (8)

converges for $|z| < |\omega|r$. Now in the derivation of (8) from (7) we can write

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| \frac{z}{hz+\omega}\right) = e^{-z}\left(1+\frac{hz}{\omega}\right)^{b}\sum_{m=0}^{n-1}\frac{C_m(b)_m}{(c)_m}\left(\frac{z}{\omega}\right)^m {}_1F_1\left(\begin{matrix} c-b \\ m+c \end{matrix}\middle| z\right) $$

$$ + \frac{\Gamma(c)}{\Gamma(b)}\left(1+\frac{hz}{\omega}\right)^{b}\left(\frac{\omega}{z}\right)^{b}\int_0^\infty R_n(t)\,dt $$

$$ R_n(t) = t^{b-1}e^{-\omega z}\sum_{m=n}^{\infty} C_m t^m E_{m+c-1}(\omega t) $$

Let ω and c be real and positive. Then

$$ |E_{m+c-1}(z)| \leqslant \frac{1}{\Gamma(c+m)} $$

By Cauchy's inequality, see Knopp (1949, p. 408), it follows from (9) that

$$ |C_m| \leqslant M\rho^m \qquad 0 < \rho < r $$

where M is a number which $|f(z)|$ never exceeds along the path $|z| = \rho$. Hence

$$ |R_n(t)| \leqslant t^{R(b)-1}\exp[-tR(\omega/z)]M\sum_{m=n}^{\infty}\frac{(t\rho)^m}{\Gamma(c+m)} $$

$$ = \frac{t^{R(b)-1}\exp[-tR(\omega/z)](t\rho)^n M}{\Gamma(c+n)}\sum_{m=0}^{\infty}\frac{(t\rho)^m}{(c+n)_m} $$

$$ |R_n(t)| < \frac{t^{R(b)-1}\exp[t/\rho - tR(\omega/z)](t\rho)^n M}{\Gamma(c+n)} $$

and so

$$ \left|\int_0^\infty R_n(t)\,dt\right| \leqslant \frac{M\Gamma(R(b)+n)\rho^{R(b)}}{\Gamma(c+n)(\rho R(\omega/z)-1)^{R(b)+n}} $$

which approaches zero as $n \to \infty$ if $\rho R(\omega/z) > 2$, i.e. if $R(z/\omega) < r/2$. Thus (8) is valid in the left part of the circle $|z/\omega| = r$. However, inside this circle, as both sides of (8) are analytic functions of z and of the parameters, this last condition as well as the conditions imposed on b, c and ω may be waived on appeal to the principle of analytic continuation.

This completes the proof of (8). Finally, to get (11) multiply both sides of the derivative of (9) with respect to z by $[1 + hz][1 + (h - 1)z]$ and equate like powers of z.

Some convergent expansions of the ${}_1F_1$ in series of Bessel functions are given in 9.4.3. If $a = -n$, and n is a large positive integer, c is fixed and $|z| < n$, an asymptotic expansion for the ${}_1F_1$ may be deduced from 7.4.6(2, 3) with $p = 0$.

For our work on rational approximations for the incomplete gamma function $\Gamma(\nu, z)$, we need some uniform representations of the confluent functions where $(c/2 - a)$ and z are large. These are given in 14.9. For other developments when a is large, and when the parameters and variable are large, see Buchholz (1953), Erdélyi *et al.* (1953), and Slater (1960).

Finally, we formally deduce a uniform asymptotic representation for the ${}_1F_1$ when c is large and $(c/2 - a)$ is bounded. With

$$y = e^{-z/2}\,{}_1F_1(a; c; z) \sim \sum_{n=0}^{\infty} d_n(z)/u^n, \qquad d_0(z) = 1, \quad u = \tfrac{1}{2}(c - 1), \tag{16}$$

$$zy'' + cy' + (k - z/4)\,y = 0, \qquad k = c/2 - a, \tag{17}$$

and the d_n's can be generated by

$$d_{n+1}(z) = -\tfrac{1}{2}z\,d_n'(z) + \tfrac{1}{2}\int_0^z (t/4 - k)\,d_n(t)\,dt. \tag{18}$$

For example,

$$\begin{aligned} d_1(z) &= \frac{z^2}{16} - \frac{kz}{2}, \qquad d_2(z) = \frac{z^4}{512} - \frac{kz^3}{32} + \frac{z^2(2k^2 - 1)}{16} + \frac{kz}{4}, \\ d_3(z) &= \frac{z^6}{24576} - \frac{kz^5}{1024} + \frac{z^4(4k^2 - 3)}{512} - \frac{k(4k^2 - 13)z^3}{192} - \frac{z^2(3k^2 - 1)}{16} - \frac{kz}{8}. \end{aligned} \tag{19}$$

If we write

$$d_n(z) = \sum_{r=1}^{2n} a_r^{(n)} z^r, \tag{20}$$

then

$$a_r^{(n+1)} = -\frac{r}{2}a_r^{(n)} - \frac{ka_{r-1}^{(n)}}{2r} + \frac{1}{8r}a_{r-2}^{(n)}, \tag{21}$$

and in particular,

$$a_{2n}^{(n)} = \frac{1}{16^n n!} \qquad a_{2n-1}^{(n)} = -\frac{k}{(2)!6^{n-1}(n-1)!}$$
$$a_2^{(n)} = \frac{(-)^n}{16}\left[\frac{2(2^{n-1}-1)k^2}{2^{n-2}} - 1\right], \qquad a_1^{(n)} = \frac{(-)^n k}{2^n} \tag{22}$$

The expressions (16)–(18) can also be deduced from some results of Olver (1954, 1956) who shows that when the second expansion in (16) is truncated after n terms, the remainder is $O(u^{-n})$ uniformly in z for $|R(z)| \leqslant b$, b fixed but arbitrary

4.9 Other Notations and Related Functions

In the literature, a notation introduced by Whittaker is often used to designate solutions of the confluent hypergeometric differential equation These functions are called Whittaker functions The connection between these functions and the ${}_1F_1$ and ψ functions are outlined below

$$M_{k,m}(z) = e^{-z/2}z^{c/2}\,{}_1F_1(a;c;z) \tag{1}$$

$$W_{k,m}(z) = W_{k,-m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}M_{k,-m}(z) \tag{2}$$

Here and throughout this discussion

$$k = c/2 - a \qquad m = (c-1)/2 \tag{3}$$

$$W_{k,m}(z) = e^{-z/2}z^{c/2}\psi(a;c;z). \tag{4}$$

$$M_{k,m}(ze^{i\pi/2}) = \exp[i\pi(m+\tfrac{1}{2})]\,M_{-k,m}(ze^{-i\pi/2}) \tag{5}$$

$$W_{k,m}(ze^{i\pi/2}) = \frac{\pi}{\sin 2m\pi}\left[-\frac{\exp[i\pi(\frac{1}{2}+m)]}{\Gamma(\frac{1}{2}-m-k)\,\Gamma(1+2m)}M_{-k,m}(ze^{-i\pi/2})\right.$$
$$\left.+\frac{\exp[i\pi(\frac{1}{2}-m)]}{\Gamma(\frac{1}{2}+m-k)\,\Gamma(1-2m)}M_{-k,-m}(ze^{-i\pi/2})\right] \tag{6}$$

$$M_{-k,m}(ze^{\pm i\pi}) = \exp[\pm i\pi(\tfrac{1}{2}+m)]\,M_{k,m}(z) \tag{7}$$

$$W_{-k,m}(ze^{\pm i\pi}) = \frac{\pi}{\sin 2m\pi}\left[-\frac{\exp[\pm i\pi(\frac{1}{2}+m)]}{\Gamma(\frac{1}{2}-m+k)\,\Gamma(1+2m)}M_{k,m}(z)\right.$$
$$\left.+\frac{\exp[\pm i\pi(\frac{1}{2}-m)]}{\Gamma(\frac{1}{2}+m+k)\,\Gamma(1-2m)}M_{k,-m}(z)\right] \tag{8}$$

Difference–differential properties follow from 4.3 We omit details A complete description is given by Slater (1960)

The confluent functions have many important special cases. The tabulation below gives a list of such transcendents and references to further information.

Function	See
Bessel functions: $J_\nu(z)$, $Y_\nu(z)$, $C_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$	6 2.7
Incomplete gamma functions and the special cases, the exponential integral, cosine- and sine-integrals, error functions, and Fresnel integrals	6 2 11, Chapter XIV
Laguerre polynomials	8.1.(33)
Hermite polynomials	8.1(34)
Coulomb wave functions	6 2.6(2–4)
Parabolic cylinder functions	6 2.6(5)

Chapter V THE GENERALIZED HYPERGEOMETRIC FUNCTION AND THE G-FUNCTION

5.1. The ${}_pF_q$ Differential Equation

In 3 2, we defined the generalized hypergeometric series

$$ {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_p)_k z^k}{(\rho_q)_k k!}, \tag{1}$$

and in Sections 3 2 3 6 we discussed many of its properties In this chapter we study other aspects of (1) and take up a generalization which is called the G-function

Consider the differential equation

$$[\delta(\delta + \rho_q - 1) - z(\delta + \alpha_p)]\, U(z) = 0, \qquad \delta = zD, \quad D = d\,dz, \tag{2}$$

where the notation is compact as in 3 2(2) That is, $(\delta + \alpha_p)$ stands for $\prod_{j=1}^{p} (\delta + \alpha_j)$, etc For connections between the δ and D operators and other related data, see 2 9

The order of (2) is $\max(p, q + 1)$ If $p < q + 1$, the singular points of (2) are at $z = 0$ and $z = \infty$, $z = 0$ is a regular singularity, $z = \infty$ an irregular singularity If $p = q + 1$, $z = 0$, $z = 1$, and $z = \infty$ are regular singularities To get a set of fundamental solutions near $z = 0$, we assume (2) has a solution of the form

$$U = \sum_{k=0}^{\infty} \tau_k z^{m+k} \tag{3}$$

Put this in (2) and get

$$\sum_{k=0}^{\infty} \tau_k\{(m + k)(m + k + \rho_q - 1)\, z^k - (m + k + \alpha_p)\, z^{k+1}\} = 0 \tag{4}$$

The coefficient of τ_0 set to zero is known as the indicial equation Let m_h, $h = 0, 1, \ldots, q$ denote its roots, which we suppose distinct Thus,

$$m_0 = 0, \qquad m_h = 1 - \rho_h, \qquad h = 1, 2, \ldots, q \tag{5}$$

If in (4), $m = m_h$ and we equate like powers of z, we find

$$v_k = \frac{(m_h + \alpha_p)_k v_0}{(m_h + 1)_k (m_h + \rho_q)_k}. \tag{6}$$

If $p \leqslant q + 1$, no ρ_h is a negative integer or zero and no two of the ρ_h's differ by an integer or zero, then the $(q + 1)$ fundamental or linearly independent solutions of (2) near $z = 0$ are proportional to

$$\begin{aligned} U_0 &= {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right), \\ U_1 &= z^{1-\rho_1}\,{}_pF_q\left(\begin{matrix}1 + \alpha_p - \rho_1\\ 2 - \rho_1, 1 + \rho_2 - \rho_1, \ldots, 1 + \rho_q - \rho_1\end{matrix}\middle| z\right), \\ U_2 &= z^{1-\rho_2}\,{}_pF_q\left(\begin{matrix}1 + \alpha_p - \rho_2\\ 1 + \rho_1 - \rho_2, 2 - \rho_2, 1 + \rho_3 - \rho_2, \ldots, 1 + \rho_q - \rho_2\end{matrix}\middle| z\right), \quad \text{etc.} \end{aligned} \tag{7}$$

If we agree to modify (2) so that there are $(q + 1)$ ρ_h's, $\rho_0, \rho_1, \ldots, \rho_q$, with the understanding that $\rho_0 = 1$, then with $p \leqslant q + 1$, the $(q + 1)$ fundamental solutions near $z = 0$ of

$$[(\delta + \rho_{q+1} - 1) - z(\delta + \alpha_p)]\, U(z) = 0 \tag{8}$$

are proportional to

$$U_h(z) = z^{1-\rho_h}\,{}_pF_q\left(\begin{matrix}1 + \alpha_p - \rho_h\\ 1 + \rho_q - \rho_h^*\end{matrix}\middle| z\right), \qquad h = 0, 1, \ldots, q, \tag{9}$$

provided that no two of the ρ_h's differ by an integer or zero. Here and elsewhere, the asterisk sign (*) indicates that the term $1 + \rho_j - \rho_h$ is omitted when $j = h$.

If we divide (8) by z and integrate, then

$$F(z) = {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) \tag{10}$$

satisfies

$$\left[(\delta + \rho_q - 1) - z\sum_{k=0}^{p-1} f_k g_{k+2}\right] F(z) - f_p \int_0^z F(t)\,dt = (\rho_q - 1),$$

$$(\rho_q - 1) = \prod_{h=1}^{q} (\rho_h - 1), \quad \text{etc.}, \quad g_k = (\delta + \alpha_k)(\delta + \alpha_{k+1}) \cdots (\delta + \alpha_p), \; g_{p+1} = 1,$$

$$f_0 = 1, \qquad f_k = (\alpha_1 - 1)(\alpha_2 - 1) \cdots (\alpha_k - 1). \tag{11}$$

If a numerator parameter is unity, say $\alpha_1 = 1$, then $F(z)$ satisfies a nonhomogeneous differential equation of order $\max(p-1, q)$ Thus,

$$[(\delta + \rho_q - 1) - z(\delta + \alpha_2) \quad (\delta + \alpha_p)]\, F(z) = (\rho_q - 1),$$

$$F(z) = {}_pF_q\left(\begin{matrix} 1, \alpha_2, \ldots, \alpha_p \\ \rho_q \end{matrix} \middle| z\right) \tag{12}$$

We also have

$$[\delta(\delta + \rho_q - 1) - z(\delta + \alpha_p)]\, A(z) = \mu(\mu + \rho_q - 1)\, z^{\mu},$$

$$A(z) = z^{\mu}\, {}_{p+1}F_{q+1}\left(\begin{matrix} 1, \mu + \alpha_p \\ \mu + 1, \mu + \rho_q \end{matrix} \middle| z\right), \tag{13}$$

$$[(\delta + \mu)(\delta + \mu + \rho_q - 1) - z(\delta + \mu + \alpha_p)]\, B(z) = \mu(\mu + \rho_q - 1),$$

$$B(z) = z^{-\mu} A(z) \tag{14}$$

If $p \geqslant q + 1$ and no two of the α_j's differ by an integer or zero, there are p fundamental solutions of (8) near $z = \infty$ proportional to

$$V_h(z) = z^{-\alpha_h}\, {}_{q+1}F_{p-1}\left(\begin{matrix} 1 + \alpha_h - \rho_q \\ 1 + \alpha_h - \alpha_p^* \end{matrix} \middle| \frac{(-)^{q+1}\ \ p}{z}\right), \qquad h = 1, 2, \ldots, p \tag{15}$$

Here again, at least one of the ρ_h terms is unity To prove this, observe that if in (8), z is replaced by $1/z$, δ must then be replaced by $-\delta$ whence

$$[(\delta + (1 - \alpha_p) - 1) - (-)^{q+1-p} z(\delta + 1 - \rho_q)]\, U(1/z) = 0, \tag{16}$$

and (15) follows by an obvious change of notation We may also deduce (15) by the usual power series approach

In the singular cases just mentioned, e g , when two or more of the ρ_h's differ by an integer or zero, independent solutions proportional to the respective U_h's do not exist In this event, we will construct solutions via limiting processes

First, we observe that $U_h(z)$ is not defined if for some $j, j \neq h$, $(1 + \rho_j - \rho_h)$ is a negative integer or zero However, the function

$$\bar{U}_h(z) = z^{1-\rho_h}\, {}_p\varphi_q\left(\begin{matrix} 1 + \alpha_p - \rho_h \\ 1 + \rho_q - \rho_h^* \end{matrix} \middle| z\right)$$

$${}_p\varphi_q(\alpha_p, \rho_q, z) = [\Gamma(\rho_q)]^{-1}\, {}_pF_q(\alpha_p, \rho_q, z) \tag{17}$$

satisfies (8) and is defined for all values of the denominator parameters

In particular, if $\rho_1 = -n$, n a positive integer or zero, and no other ρ_h is a negative integer or zero, then

$$ {}_p\varphi_q(\alpha_p\,;\rho_q\,;z) = \frac{(\alpha_p)_{n+1}z^{n+1}}{(\beta_q)_{n+1}\Gamma(\beta_q)}\,{}_pF_q(\alpha_p+n+1;\beta_q+n+1;z), $$

$$ \beta_1 = 1, \qquad \beta_j = \rho_j \qquad \text{for} \quad j = 2, 3,\ldots, q, \tag{18} $$

which is $U_1(z)$ except for a constant factor. Thus no new solution obtains.

Now consider the G-function defined by

$$ G^{m+1,n}_{p,q+1}\left(z\exp[-i\pi(m+n+1-p)]\,\middle|\,\begin{matrix}1-\alpha_1,\ldots,1-\alpha_p\\ 1-\rho_0,\ldots,1-\rho_q\end{matrix}\right) $$

$$ = G^{m+1,n}_{p,q+1}\left(z\exp[-i\pi(m+n+1-p)]\,\middle|\,\begin{matrix}1-\alpha_p\\ 1-\rho_q\end{matrix}\right) = \sum_{h=0}^{m} A_h^{(m)}U_h(z) $$

$$ = \sum_{h=0}^{m}\frac{(z\exp[-i\pi(m+n+1-p)])^{(1-\rho_h)}\prod_{j=0}^{m}\Gamma(\rho_h-\rho_j)^*\prod_{j=1}^{n}\Gamma(1+\alpha_j-\rho_h)}{\prod_{j=m+1}^{q}\Gamma(1+\rho_j-\rho_h)\prod_{j=n+1}^{p}\Gamma(\rho_h-\alpha_j)} $$

$$ \times\,{}_pF_q\left(\begin{matrix}1+\alpha_p-\rho_h\\ 1+\rho_q-\rho_h^*\end{matrix}\,\middle|\,z\right), $$

$$ 0\leqslant m\leqslant q,\quad 0\leqslant n\leqslant p;\quad p<q+1 \quad\text{or}\quad p=q+1 \quad\text{and}\quad |z|<1, \tag{19} $$

where $U_h(z)$ is given by (9) and the definition of $A_h^{(m)}$ is obvious. Note that if in (19), the G-function is defined without any reference to the functions $U_h(z)$, then ρ_0 need not be unity.

A more detailed analysis of the G-function will be given later, see 5.2. However, a special case has already been encountered. For if $m = n = q = 1$ and $p = 2$, then [see 3.10(1)],

$$ V(z) = \frac{\Gamma(1+a_1-a_2)}{\Gamma(1+a_1-b_0)\,\Gamma(1+a_1-b_1)}\,G^{2,1}_{2,2}\left(ze^{-i\pi}\,\middle|\,\begin{matrix}1-a_1,\,1-a_2\\ 1-b_0,\,1-b_1\end{matrix}\right). \tag{20} $$

Indeed, our discussion concerning independent solutions of (8) when two ρ_h's differ by an integer or zero is akin to that given for the ${}_2F_1$. For each $m = 0, 1,\ldots, q$, (19) gives linearly independent solutions of (8) provided that none of the ρ_h's differ by an integer or zero, and that $(1+\alpha_j-\rho_h)$, $j = 1, 2,\ldots, n$, is not a negative integer or zero. The latter requirement is not restrictive insofar as solutions of (8) are concerned, since solutions are only known to within an arbitrary constant. Further, n is arbitrary, so that we could arrange the parameters to have $\Gamma(\rho_i-\alpha_k)$ appear in the denominator of the right-hand side of (19) if $(1+\alpha_k-\rho_i)$ is a negative integer or zero. In this connection, we should note that the G-function is symmetric in the parameters: $\alpha_1,\ldots,\alpha_n$;

$\alpha_{n+1}, \ldots, \alpha_p, \rho_0, \ldots, \rho_m$, and $\rho_{m+1}, \ldots, \rho_q$. Thus a shuffling of the parameters on the righthand side of (19) may necessitate a change in the arrangement of the parameters of the G function on the left. But as previously remarked, this does not affect our approach to writing solutions which satisfy (8).

If $m \geqslant 1$, $A_0^{(m)}U_0(z)$ and $A_0^{(m)}U_0(z) + A_1^{(m)}U_1(z)$ with $\rho_1 - \rho_0 = s + \epsilon$, s a positive integer or zero, are linearly independent solutions of (8). If $\epsilon = 0$, we invoke L'Hospital's theorem to get a solution independent of $U_0(z)$ by evaluating

$$\Pi_1(z) = \lim_{\epsilon \to 0} \{A_0^{(m)}U_0(z) + A_1^{(m)}U_1(z)\} \tag{21}$$

We prove that

$$\begin{aligned}
\Pi_1(z) &= T_{p\,q+1}^{m+1\,n}\left(z \exp[-i\pi(m+n+1-p)] \,\middle|\, \begin{matrix} 1, 1+\rho_0-\rho_q \\ 1+\rho_0-\alpha_p \end{matrix}\right) \\
&+ \frac{(\quad)^{s-1}(z \exp[-i\pi(m+n+1-p)])^{1-\rho_0} \prod_{j=2}^{m} \Gamma(\rho_0-\rho_j) \prod_{j=1}^{n} \Gamma(1+\alpha_j-\rho_0)}{s! \prod_{j=m+1}^{q} \Gamma(1+\rho_j-\rho_0) \prod_{j=n+1}^{p} \Gamma(\rho_0-\alpha_j)} \\
&\times \Big\{\Big[\gamma + \ln(z \exp[-i\pi(m+n+1-p)]) - \sum_{j=2}^{m} \psi(\rho_0-\rho_j) - \sum_{j=m+1}^{q} \psi(1+\rho_j-\rho_0) \\
&+ \sum_{j=1}^{n} \psi(1+\alpha_j-\rho_0) + \sum_{j=n+1}^{p} \psi(\rho_0-\alpha_j) - \psi(1+s)\Big] {}_pF_q\left(\begin{matrix} 1+\alpha_p-\rho_0 \\ 1+\rho_q-\rho_0^* \end{matrix} \,\middle|\, z\right) \\
&+ {}_pF_{q+1}\left(\begin{matrix} 1+\alpha_p-\rho_0 \\ 1+\rho_q-\rho_0^* \end{matrix} \,\middle|\, \begin{matrix} 1+\alpha_p-\rho_0 \\ 1+\rho_q-\rho_0 \end{matrix} \,\middle|\, z\right)\Big\}, \qquad \rho_1 = \rho_0 + s
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
&T_{p\,q}^{m\,n}\left(z \,\middle|\, \begin{matrix} 1, 1+\tau_q-\tau_2 \\ 1+\sigma_p-\tau_2 \end{matrix}\right) \\
&= T_{p\,q}^{m\,n}\left(z \,\middle|\, \begin{matrix} 1, 1-s, 1+\tau_2-\tau_2, \ldots, 1+\tau_q-\tau_2 \\ 1+\sigma_1-\tau_2, \ldots, 1+\sigma_p-\tau_2 \end{matrix}\right) \\
&= z^{\tau_1} \frac{\prod_{j=2}^{m} \Gamma(\tau_j-\tau_1) \prod_{j=1}^{n} \Gamma(1+\tau_1-\sigma_j)}{\prod_{j=m+1}^{q} \Gamma(1+\tau_1-\tau_j) \prod_{j=n+1}^{p} \Gamma(\sigma_j-\tau_1)} \\
&\quad \times \sum_{k=0}^{s-1} \frac{(1+\tau_1-\sigma_p)_k (s-1-k)!(\quad)^{k(m+n+1-p)} z^k}{(1+\tau_1-\tau_2)_k \quad (1+\tau_1-\tau_q)_k k!} \\
&= \frac{z^{\tau_1}\{z \exp[i\pi(m+n+1-p)]\}^{s-1} \pi^{m+n-p-2} \prod_{j=1}^{p} \Gamma(\tau_2-\sigma_j) \prod_{j=n+1}^{p} \sin \pi(\sigma_j-\tau_1)}{(s-1)! \prod_{j=2}^{q} \Gamma(\tau_2-\tau_j) \prod_{j=2}^{m} \sin \pi(\tau_j-\tau_1)} \\
&\quad \times {}_{q+1}F_p\left(\begin{matrix} 1, 1+\tau_q-\tau_2 \\ 1+\sigma_p-\tau_2 \end{matrix} \,\middle|\, \frac{(-)^{p-m-n}}{z}\right) \qquad \tau_2 - \tau_1 = s
\end{aligned} \tag{23}$$

and

$$ {}_{p}^{u}F_{q}^{v}\left(\begin{matrix}\sigma_p \\ \tau_q\end{matrix}\middle\| \begin{matrix}\gamma_u \\ \delta_v\end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(\sigma_p)_k z^k}{(\tau_q)_k k!} \{\psi(\gamma_u + k) - \psi(\gamma_u) - \psi(\delta_v + k) + \psi(\delta_v)\}. \tag{24} $$

In the latter, $\psi(\gamma_u)$ is short for $\sum_{j=1}^{u} \psi(\gamma_j)$, etc.

Note that in the ${}_pF_q^{q+1}$ symbol in (22), $1 + \rho_q - \rho_0$ stands for $1, 1 + s, 1 + \rho_2 - \rho_0, \ldots, 1 + \rho_q - \rho_0$, and $1 + \rho_q - \rho_0^*$ is the same with 1 omitted. Again in (23), $1 + \tau_q - \tau_2$ stands for $1 - s, 1, 1 + \tau_3 - \tau_2, \ldots, 1 + \tau_q - \tau_2$. To get the polynomial expansion for $T_{p,q+1}^{m+1,n}$ in (22) from that of $T_{p,q}^{m,n}$ in (23), in the latter replace m by $m + 1$, q by $q + 1$, σ_h by $1 - \alpha_h$, τ_1 by $1 - \rho_1$, τ_2 by $1 - \rho_0$, and τ_{h+1} by $1 - \rho_h$ for $h = 2, 3, \ldots, q$. Then replace z by $z \exp[-i\pi(m + n + 1 - p)]$.

For the proof of (22) we write $A_1^{(m)} U_1(z)$ in the form $C + D$ where $C = \sum_{k=0}^{s-1}$ and $D = \sum_{k=s}^{\infty}$. Then $\lim_{\epsilon \to 0} C$ is given by (23) and

$$ D = \frac{\left[\begin{matrix}(-)^s(z \exp[-i\pi(m + n + 1 - p)])^{(1-\rho_1+s)}\Gamma(1 + \epsilon) \\ \times \prod_{j=2}^{m} \Gamma(\rho_1 - \rho_j - s) \prod_{j=1}^{n} \Gamma(1 + \alpha_j - \rho_1 + s)\end{matrix}\right]}{\epsilon\Gamma(1 + s) \prod_{j=m+1}^{q} \Gamma(1 + \rho_j - \rho_1 + s) \prod_{j=n+1}^{p} \Gamma(\rho_1 - \alpha_j - s)} $$

$$ \times {}_{p+1}F_{q+1}\left(\begin{matrix}1, 1 + \alpha_p - \rho_1 + s \\ 1 - \epsilon, 1 + s, 1 + \rho_2 - \rho_1 + s, \ldots, 1 + \rho_q - \rho_1 + s\end{matrix}\middle| z\right). \tag{25} $$

Also,

$$ A_0^{(m)} U_0(z) $$

$$ = \frac{\left[\begin{matrix}(-)^{s+1}(z \exp[-i\pi(m + n + 1 - p)])^{(1-\rho_1+s+\epsilon)}\Gamma(1 - \epsilon)\Gamma(1 + \epsilon) \\ \times \prod_{j=2}^{m} \Gamma(\rho_1 - \rho_j - s - \epsilon) \prod_{j=1}^{n} (1 + \alpha_j - \rho_0 + s + \epsilon)\end{matrix}\right]}{\epsilon\Gamma(1 + s + \epsilon) \prod_{j=m+1}^{q} \Gamma(1 + \rho_j - \rho_1 + s + \epsilon) \prod_{j=n+1}^{p} \Gamma(\rho_1 - \alpha_j - s - \epsilon)} $$

$$ \times {}_pF_q\left(\begin{matrix}1 + \alpha_p - \rho_1 + s + \epsilon \\ 1 + s + \epsilon, 1 + \rho_2 - \rho_1 + s + \epsilon, \ldots, 1 + \rho_q - \rho_1 + s + \epsilon\end{matrix}\middle| z\right). \tag{26} $$

Now evaluate $\lim_{\epsilon \to 0}\{D + A_0^{(m)} U_0(z)\}$ and (22) readily follows.

If another pair of the ρ_j's differ by a positive integer or zero, say $\rho_2 - \rho_3 = r$, but ρ_2 and ρ_1 do not differ by an integer or zero, then a solution independent of $U_3(z)$ follows on application of the above analysis to $A_2^{(m)} U_2(z) + A_3^{(m)} U_3(z)$. If three or more of the ρ_j's differ by an integer or zero, the previous analysis can be readily extended to get linearly independent solutions. Later, we define the G-function by a Mellin–Barnes integral and after the manner of deriving 3.10(13) as in 3.10(14, 15), we can deduce (22). This same type of analysis may be used to get a representation when three or more of the ρ_j's differ by an integer or zero. Equation (22) has been previously given by Luke (1962a).

There on p 15, in (2), for $-\psi(1+\beta_q+k)+\psi(1+\beta_q)$ read the same with β_q replaced by δ_s See also MacRobert (1961, 1962b) for an analysis when two or more of the ρ_j's differ by an integer or zero

We now consider (15) If $\alpha_2-\alpha_1=s$, s a positive integer or zero, and no other α_h differs from α_1 by an integer or zero, then $V_2(z)$ is defined and $V_1(z)$ is not defined unless $s=0$, in which case $V_1(z)=V_2(z)$ If $s>0$, a well defined solution may be written using the ${}_p\varphi_q$ symbol [see the remarks around (17)] However, this leads to a solution proportional to $V_2(z)$ For our later work, it is convenient to write

$$G^{p,1}_{q+1,p}\left((ze^{-i\pi})^{-1}\middle|\begin{matrix}\rho_0 & \rho_1, & \ldots, \rho_q\\ & \alpha_p & \end{matrix}\right)=\sum_{h=1}^{p}B_h^{(p)}V_h(z)$$

$$=\frac{\sum_{h=1}^{p}\Gamma(\alpha_h)\prod_{j=1}^{p}\Gamma(\alpha_j-\alpha_h)^*(ze^{-i\pi})^{-\alpha_h}}{\prod_{j=1}^{q}\Gamma(\rho_j-\alpha_h)}$$

$$\times\,{}_{q+1}F_{p-1}\left(\begin{matrix}1+\alpha_h-\rho_q\\ 1+\alpha_h-\alpha_p^*\end{matrix}\middle|\frac{(-)^{q+1-p}}{z}\right),$$

$$q+1<p \quad\text{or}\quad q+1=p \quad\text{and}\quad |z|>1, \tag{27}$$

where $V_h(z)$ is given by (15) in which event we take $\rho_0=1$ The definition of $B_h^{(p)}$ is obvious If we omit reference to the functions $V_h(z)$, then (27) is valid without $\rho_0=1$, see 5 2(11) with an appropriate change of notation If $\rho_0=1$, the above relation is notated $L_{p,q}(ze^{-i\pi})$ [see 5 11 1(7)]

To derive a solution independent of $V_2(z)$, we set $\alpha_2-\alpha_1=s+\epsilon$ and evaluate

$$X(z)=\lim_{\epsilon\to 0}\{B_1^{(p)}V_1(z)+B_2^{(p)}V_2(z)\} \tag{28}$$

After the manner of deriving (22), we find that

$$X(z)=T^{p,1}_{q+1,p}\left((ze^{-i\pi})^{-1}\middle|\begin{matrix}1 & 1+\alpha_p-\alpha_2\\ & 1+\rho_q-\alpha_2\end{matrix}\right)$$

$$+\frac{(-)^{s+1}(ze^{-i\pi})^{-\alpha_2}\Gamma(\alpha_2)\prod_{j=3}^{p}\Gamma(\alpha_j-\alpha_2)}{s!\prod_{j=1}^{q}\Gamma(\rho_j-\alpha_2)}$$

$$\times\left\{\left[\gamma+\ln(ze^{-i\pi})^{-1}+\psi(\alpha_2)-\sum_{j=3}^{p}\psi(\alpha_j-\alpha_2)+\sum_{j=1}^{q}\psi(\rho_j-\alpha_2)-\psi(1+s)\right]\right.$$

$$\times\,{}_{q+1}F_{p-1}\left(\begin{matrix}1+\alpha_2-\rho_q\\ 1+\alpha_2-\alpha_p^*\end{matrix}\middle|\frac{(-)^{q-p+1}}{z}\right)$$

$$\left.+\,{}^{q+1}_{q+1}F^{p}_{p-1}\left(\begin{matrix}1+\alpha_2-\rho_q\\ 1+\alpha_2-\alpha_p^*\end{matrix}\middle\|\begin{matrix}1+\alpha_2-\rho_q\\ 1+\alpha_2-\alpha_p\end{matrix}\middle|\frac{(-)^{q-p+1}}{z}\right)\right\}$$

$$\alpha_2-\alpha_1=s \tag{29}$$

where the $T^{m,n}_{p,q}$ and ${}^{u}_{p}F^{v}_{q}$ symbols are given by (23) and (24), respectively. For convenience the polynomial portion of (26) may be expressed in the form

$$T^{p,1}_{q+1,p}\left((ze^{-i\pi})^{-1}\,\middle|\,\begin{matrix}1,\,1+\alpha_p-\alpha_2\\ 1+\rho_q-\alpha_2\end{matrix}\right)$$

$$=(ze^{-i\pi})^{-\alpha_1}\sum_{k=0}^{s-1}\frac{(s-1-k)!\,\Gamma(\alpha_1+k)\prod_{j=3}^{p}\Gamma(\alpha_j-\alpha_1-k)}{k!\prod_{j=1}^{q}\Gamma(\rho_q-\alpha_1-k)\,z^k}$$

$$=\frac{e^{i\pi\alpha_1}z^{1-\alpha_2}\Gamma(\alpha_2-1)\prod_{j=3}^{p}\Gamma(1+\alpha_j-\alpha_2)}{(s-1)!\prod_{j=1}^{q}\Gamma(1+\rho_j-\alpha_2)}\,{}_{p+1}F_{q+1}\left(\begin{matrix}1,\,1+\alpha_p-\alpha_2\\ 2-\alpha_2,\,1+\rho_q-\alpha_2\end{matrix}\,\middle|\,z\right). \tag{30}$$

Clearly (29) may be deduced from (22) with a change of notation.

As previously remarked, the point $z=1$ is a regular singularity of (2) when $p=q+1$. For fundamental solutions near this singularity, we refer the reader to the excellent work of Nörlund (1955).

5.2. The G-Function

In 3.6(28) and 4.2(4) we gave an integral representation of the Mellin–Barnes type for the ${}_2F_1$ and ${}_1F_1$ functions, respectively. With these as patterns, we could write down and prove a similar representation for the ${}_pF_q$. It is more convenient, however, to deal with a function which includes the ${}_pF_q$ as a special case. Such a transcendent is the G-function which we now define as

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)=G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)$$

$$=(2\pi i)^{-1}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\,z^s\,ds. \tag{1}$$

Where no confusion can result, we often refer to the latter as $G^{m,n}_{p,q}(z)$. Here an empty product is interpreted as unity, $0\leqslant m\leqslant q$, $0\leqslant n\leqslant p$, and the parameters a_h and b_h are such that no pole of $\Gamma(b_j-s)$, $j=1,2,\ldots,m$, coincides with any pole of $\Gamma(1-a_k+s)$, $k=1,2,\ldots,n$. Thus (a_k-b_j) is not a positive integer. We retain these assumptions throughout. Also $z\neq 0$.

There are three different paths L of integration

L goes from $-i\infty$ to $+i\infty$ so that all poles of $\Gamma(b_j - s)$, $j = 1, 2, \ldots, m$, lie to the right of the path and all poles of $\Gamma(1 - a_k + s)$, $k = 1, 2, \ldots, n$ lie to the left of the path For the integral to converge we need $\delta = m + n - \frac{1}{2}(p + q) > 0$, $|\arg z| < \delta\pi$ If $|\arg z| = \delta\pi$, $\delta \geqslant 0$, the integral converges absolutely when $p = q$ if $R(\nu) < -1$, and when $p \neq q$, if with $s = \sigma + i\tau$, σ and τ real, σ is chosen so that for $\tau \to \pm\infty$, (2)

$$(q - p)\sigma > R(\nu) + 1 - \tfrac{1}{2}(q - p),$$

where ν is given by (6) below

L is a loop beginning and ending at $+\infty$ and encircling all poles of $\Gamma(b_j - s)$ $j = 1, 2, \ldots, m$, once in the negative direction, but none of the poles of $\Gamma(1 - a_k + s)$, $k = 1, 2, \ldots, n$ The integral converges if $q \geqslant 1$ and either $p < q$ or $p = q$ and $|z| < 1$ (3)

L is a loop beginning and ending at $-\infty$ and encircling all poles of $\Gamma(1 - a_k + s)$, $k = 1, 2, \ldots, n$, once in the positive direction, but none of the poles of $\Gamma(b_j - s)$ $j = 1, 2, \ldots, m$ The integral converges if $p \geqslant 1$ and either $p > q$ or $p = q$ and $|z| > 1$ (4)

For later considerations, it is informative to verify the statements given in (2) Let

$$B(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s) z^s}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)}, \tag{5}$$

and let $s = \sigma + i\tau$, σ and τ real, on the path of integration Employing 2 11(12), we find

$$|B(s)| \leqslant C \exp[-(\delta\pi|\tau| + \tau \arg z)]\, |\tau|^{\theta}[1 + \eta(\tau)\, O(\tau^{-1})],$$

$$C = (2\pi)^{\delta} \exp\left[\frac{\pi}{2}\left\{\sum_{j=1}^{q} |I(b_j)| + \sum_{j=1}^{p} |I(a_j)|\right\}\right],$$

$$\theta = R(\nu) + (\sigma + \tfrac{1}{2})(p - q) \qquad \nu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j,$$

$$\delta = m + n - \tfrac{1}{2}(p + q) \tag{6}$$

where $\eta(\tau)$ is bounded for all τ If $\delta > 0$ and $|\arg z| < \delta\pi$, $B(s)$ is of exponential decay for large τ on the path of integration and so the

integral converges. If $|\arg z| = \delta\pi$, $\delta \geqslant 0$, the integral will converge absolutely provided that for τ sufficiently large, $\theta < -1$ and this leads to the statement (2).

It is supposed that the parameters a_h, b_h, and the variable z are such that at least one of the definitions (2)–(4) makes sense. Where more than one of the definitions has meaning, they lead to the same result so that no confusion arises.

If we use (3), the integral can be evaluated as a sum of residues. If no two of the b_h terms, $h = 1, 2,..., m$, differ by an integer or zero, all poles are simple, and

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = \sum_{h=1}^{m} \frac{\prod_{j=1}^{m} \Gamma(b_j - b_h)^* \prod_{j=1}^{n} \Gamma(1 + b_h - a_j) z^{b_h}}{\prod_{j=m+1}^{q} \Gamma(1 + b_h - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - b_h)}$$

$$\times {}_pF_{q-1}\left(\begin{matrix} 1 + b_h - a_p \\ 1 + b_h - b_q^* \end{matrix} \,\middle|\, (-)^{p-m-n} z\right),$$

$$p < q \quad \text{or} \quad p = q \quad \text{and} \quad |z| < 1. \tag{7}$$

If $m = 0$, and we use (3), the integrand is analytic on and within the contour and so

$$G_{p,q}^{0,n}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = 0. \tag{8}$$

It is clear that the G-function is a many valued function of z with a branch point at the origin. For further discussion of the singularities of the G-function, see the remarks surrounding 5.3(1) and 5.8(1).

If in (7), we replace m by $m + 1$, q by $q + 1$, and a_h by $1 - \alpha_h$, b_h by $1 - \rho_h$ and z by $z \exp\{-i\pi(m + n + 1 - p)\}$, we get 5.1(19). Again replace p by $q + 1$ and a_h by ρ_{h-1}, $h = 1,..., q + 1$, with $\rho_0 = 1$, replace q by p and b_h by α_h, $h = 1, 2,..., p$, take $m = p$, $n = 1$, and replace z by $(ze^{-i\pi})^{-1}$. We then get 5.1(27).

If only two of the b_h's differ by an integer or zero, then as in our previous studies, we can use L'Hospital's theorem on the pertinent portion of the right-hand side of (7) to get a series representation for the G-function on the left. The final result can be deduced from 5.1(22) by making appropriate substitutions for the parameters and variable previously noted. Suppose that $b_2 - b_1 = s$, s a positive integer or zero. Write (7) in the form

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = \sum_{h=1}^{m} C_h^{(m)}, \qquad m \geqslant 2, \tag{9}$$

where the meaning of $C_h^{(m)}$ is evident Then

$$G_{p\,q}^{m\,n}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) = \sum_{h=3}^{m} C_h^{(m)} + T_{p\,q}^{m\,n}\left(z\,\middle|\,\begin{matrix}1 & 1+b_q-b_2\\ & 1+a_p-b_2\end{matrix}\right)$$

$$+ \frac{(-)^{s+1} z^{b_2} \prod_{j=3}^{m} \Gamma(b_j - b_2) \prod_{j=1}^{n} \Gamma(1 + b_2 - a_j)}{s! \prod_{j=m+1}^{q} \Gamma(1 + b_2 - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - b_2)}$$

$$\times \left[\left\{(\gamma + \ln z) - \sum_{j=3}^{m} \psi(b_j - b_2) + \sum_{j=1}^{n} \psi(1 + b_2 - a_j)\right.\right.$$

$$\left. - \sum_{j=m+1}^{q} \psi(1 + b_2 - b_j) + \sum_{j=n+1}^{p} \psi(a_j - b_2) - \psi(1 + s)\right\}$$

$$\times {}_pF_{q-1}\left(\begin{matrix}1 + b_2 - a_p\\ 1 + b_2 - b_q^*\end{matrix}\,\middle|\,(-)^{p-m-n} z\right)$$

$$\left. + {}_pF_{q-1}^{q}\left(\begin{matrix}1 + b_2 - a_p\\ 1 + b_2 - b_q^*\end{matrix}\,\middle\|\,\begin{matrix}1 + b_2 - a_p\\ 1 + b_2 - b_q\end{matrix}\,\middle|\,(-)^{p-m-n} z\right)\right],$$

$$b_2 = b_1 + s, \tag{10}$$

where the $T_{p\,q}^{m\,n}$ and ${}_pF_{q-1}^{q}$ notations are defined by 5 1(23) and 5 1(24), respectively Equation (10) can also be deduced from the contour integral (1) See the remarks following 5 1(26)

If no two of the a_h terms, $h = 1, \ldots, n$, differ by an integer or zero, then use of (4) leads to

$$G_{p\,q}^{m\,n}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) = \sum_{h=1}^{n} \frac{\prod_{j=1}^{n} \Gamma(a_h - a_j)^* \prod_{j=1}^{m} \Gamma(b_j - a_h + 1)\, z^{a_h - 1}}{\prod_{j=n+1}^{p} \Gamma(1 + a_j - a_h) \prod_{j=m+1}^{q} \Gamma(a_h - b_j)}$$

$$\times {}_qF_{p-1}\left(\begin{matrix}1 + b_q - a_h\\ 1 + a_p - a_h^*\end{matrix}\,\middle|\,\frac{(-)^{q-m-n}}{z}\right),$$

$$q < p \quad \text{or} \quad q = p \quad \text{and} \quad |z| > 1 \tag{11}$$

It is clear from (7) and (11) that the ${}_pF_q$ can be expressed as a G-function We have the following relations

$$G_{p\,q}^{1\,n}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) = \frac{\prod_{j=1}^{n} \Gamma(1 + b_1 - a_j)\, z^{b_1}}{\prod_{j=2}^{q} \Gamma(1 + b_1 - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - b_1)}$$

$$\times {}_pF_{q-1}\left(\begin{matrix}1 + b_1 - a_p\\ 1 + b_1 - b_q^*\end{matrix}\,\middle|\,(-)^{p-1-n} z\right),$$

$$p < q \quad \text{or} \quad p = q \quad \text{and} \quad |z| < 1, \tag{12}$$

$$
{}_pF_q\left(\begin{matrix}1+\alpha_p-\rho_0\\ 1+\rho_q-\rho_0^*\end{matrix}\middle|\, z\right)=\frac{\left[\begin{matrix}\prod_{j=1}^{q}\Gamma(1+\rho_j-\rho_0)\prod_{j=n+1}^{p}\Gamma(\rho_0-\alpha_j)\\ \times\{z\exp[-i\pi(n+1-p)]\}^{(\rho_0-1)}\end{matrix}\right]}{\prod_{j=1}^{n}\Gamma(1+\alpha_j-\rho_0)}
$$

$$
\times G^{1,n}_{p,q+1}\left(z\exp[-i\pi(n+1-p)]\,\middle|\,\begin{matrix}1-\alpha_p\\ 1-\rho_0,1-\rho_1,\ldots,1-\rho_q\end{matrix}\right),
$$

$$
p\leqslant q \quad\text{or}\quad p=q+1 \quad\text{and}\quad |z|<1; \tag{13}
$$

$$
{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\, z\right)=\frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)}G^{1,p}_{p,q+1}\left(-z\,\middle|\,\begin{matrix}1-\alpha_p\\ 0,1-\rho_q\end{matrix}\right),
$$

$$
p\leqslant q \quad\text{or}\quad p=q+1 \quad\text{and}\quad |z|<1. \tag{14}
$$

We remind the reader that in (13), ρ_0 need not be unity.

$$
G^{m,1}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)=\frac{\prod_{j=1}^{m}\Gamma(b_j-a_1+1)\,z^{a_1-1}}{\prod_{j=2}^{p}\Gamma(1+a_j-a_1)\prod_{j=m+1}^{q}\Gamma(a_1-b_j)}
$$

$$
\times {}_qF_{p-1}\left(\begin{matrix}1+b_q-a_1\\ 1+a_p-a_1^*\end{matrix}\middle|\,\frac{(-)^{q-m-1}}{z}\right),
$$

$$
q<p \quad\text{or}\quad q=p \quad\text{and}\quad |z|>1. \tag{15}
$$

$$
{}_pF_q\left(\begin{matrix}1+\alpha_p-\rho_0\\ 1+\rho_q-\rho_0^*\end{matrix}\middle|\, z\right)=\frac{\left[\begin{matrix}\prod_{j=1}^{q}\Gamma(1+\rho_j-\rho_0)\prod_{j=n+1}^{p}\Gamma(\rho_0-\alpha_j)\\ \times\{z\exp[-i\pi(n+1-p)]\}^{(\rho_0-1)}\end{matrix}\right]}{\prod_{j=1}^{n}\Gamma(1+\alpha_j-\rho_0)}
$$

$$
\times G^{n,1}_{q+1,p}\left(z^{-1}\exp[i\pi(n+1-p)]\,\middle|\,\begin{matrix}\rho_0,\rho_1,\ldots,\rho_q\\ \alpha_p\end{matrix}\right),
$$

$$
p\leqslant q \quad\text{or}\quad p=q+1 \quad\text{and}\quad |z|<1. \tag{16}
$$

$$
{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\, z\right)=\frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)}G^{p,1}_{q+1,p}\left(-z^{-1}\,\middle|\,\begin{matrix}1,\rho_q\\ \alpha_p\end{matrix}\right),
$$

$$
p\leqslant q \quad\text{or}\quad p=q+1 \quad\text{and}\quad |z|<1. \tag{17}
$$

The combination (7) and (12) yields the expansion formula

$$
G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)=\sum_{h=1}^{m}\frac{\left[\begin{matrix}\prod_{j=1}^{m}\{\Gamma(b_j-b_h)^*\,\Gamma(1+b_h-b_j)^*\}\\ \times\exp[-i\pi b_h(p+1-m-n)]\end{matrix}\right]}{\prod_{j=n+1}^{p}\{\Gamma(a_j-b_h)\,\Gamma(1+b_h-a_j)\}}
$$

$$
\times G^{1,p}_{p,q}\left((-)^{p+1-m-n}z\,\middle|\,\begin{matrix}a_p\\ b_h,b_q^*\end{matrix}\right). \tag{18}
$$

Similarly, from (11) and (15)

$$G^{m\,n}_{p\,q}\left(z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)=\sum_{h=1}^{n}\frac{\left[\prod_{j=1}^{n}\{\Gamma(a_h-a_j)^*\,\Gamma(1+a_j-a_h)^*\times\exp[-i\pi(a_h-1)(q+1-m-n)]\right]}{\prod_{j=m+1}^{q}\{\Gamma(a_h-b_j)\,\Gamma(1+b_j-a_h)\}}$$

$$\times\,G^{q\,1}_{p\,q}\left((-)^{q+1-m}\,{}^{n}z\left|\begin{matrix}a_h,\,a_p^*\\ b_q\end{matrix}\right.\right)\tag{19}$$

For generalizations of (18), (19) and other series, see 5 9 3

Originally the G-function was defined by Meijer (1936) by the series (7) Later, see Meijer (1941a, p 83, 1946), the series definition was replaced by one in terms of the Mellin–Barnes integral (1) where the path L is as defined in (2) and (3), respectively The complete definition (1)–(4), except for the discussion involving $|\arg z|=\delta\pi$, $\delta\geqslant 0$, in (2), is due to Erdélyi *et al* (1953, Vol 1, p 207)

The E-function introduced by MacRobert (1937, 1938) can be defined by

$$E(a_1,a_2,\ \ ,a_p,b_1,b_2,\ \ ,b_q,z)=E(a_p,b_q,z)=G^{p\,1}_{q+1\,p}\left(z\left|\begin{matrix}1,b_q\\ a_p\end{matrix}\right.\right)\tag{20}$$

Both MacRobert's E-function and Meijer's G-function arose from an attempt to give meaning to the ${}_pF_q$ symbol when $p>q+1$

5 3. Analytic Continuation of $G^{m,n}_{p,p}(z)$

It is readily shown from 5 2(1) that

$$G^{m\,n}_{p\,q}\left(z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)=G^{n\,m}_{q\,p}\left(z^{-1}\left|\begin{matrix}1-b_q\\ 1-a_p\end{matrix}\right.\right),\qquad \arg\left(\frac{1}{z}\right)=-\arg z\tag{1}$$

This is an important relation, for in the discussion of the G-function we can without loss of generality suppose that $p\leqslant q$ If $p=q$ in 5 2(1) and L is defined as in 5 2(3), then we require $0<|z|<1$, whereas if L is given by 5 2(4), $|z|>1$ When L is the path in 5 2(2), we have a representation valid for all z, $z\neq 0$, provided $m+n\geqslant p+1$ and $|\arg z|<(m+n-p)\pi$ When $|z|<1$, we can, without altering the value of $G^{m\,n}_{p\,p}(z)$, bend the contour in (2) around so that it coincides with the contour in (3) From 5 8(1), $G^{m\,n}_{p\,p}(z)$ satisfies a linear differential equation of order p and every finite point in the z-plane is an ordinary point save $z=0$ and $z=(-)^{m+n-p}$ which are regular singularities

To define the $G_{p,p}^{m,n}(z)$ function outside the unit circle, we introduce a cross cut in the z-plane along the straight line from

$$(-)^{m+n-p} \quad \text{to} \quad (-)^{m+n-p}(1 + \infty e^{i\mu}), \qquad -\pi/2 \leqslant \mu \leqslant \pi/2. \tag{2}$$

In (2), we most always take $\mu = 0$. Then in the cut plane $G_{p,p}^{m,n}(z)$ has no singularity except for the branch point at $z = 0$. If $m + n \geqslant p + 2$, the sector $|\arg z| < (m + n - p)\pi$ contains the point $(-)^{m+n-p}$. But in this situation, as noted above, $G_{p,p}^{m,n}(z)$ is regular at $z = (-)^{m+n-p}$. Thus, if $m + n \geqslant p + 2$ and $|\arg z| < (m + n - p)\pi$, the cross cut is superfluous. We have, therefore, proved that if $m + n \geqslant p + 1$ and $|\arg z| < (m + n - p)\pi$, then $G_{p,p}^{m,n}(z)$ can be continued analytically from inside the unit disc with center at the origin to the outside of this disc by means of the expansion 5.2(11) with $q = p$. It readily follows that

$$\begin{aligned} {}_{p+1}F_p\left(\begin{matrix} a_{p+1} \\ b_p \end{matrix}\middle| z\right) &= \frac{\Gamma(b_p)}{\Gamma(a_{p+1})} \sum_{h=1}^{p+1} \Gamma(a_h) \frac{\prod_{j=1}^{p+1} \Gamma(a_j - a_h)^*(z^{-1}e^{i\pi})^{a_h}}{\prod_{j=1}^{p} \Gamma(b_j - a_h)} \\ &\quad \times {}_{p+1}F_p\left(\begin{matrix} 1 + a_h - b_p, a_h \\ 1 + a_h - a_{p+1}^* \end{matrix}\middle| z^{-1}\right), \\ &\quad 0 < \arg z < 2\pi, \end{aligned} \tag{3}$$

and this furnishes the analytic continuation of the ${}_{p+1}F_p$ on the left from inside the unit circle to the outside of this circle. We have therefore completely generalized the ${}_2F_1$ results of Chapter III with respect to solutions of 5.1(2) about the singular points $z = 0$ and $z = \infty$, and the connection between these solutions. In the ${}_{p+1}F_p$ situation for $p > 1$, the solutions about the singular point $z = 1$ are not of hypergeometric type and so are rather complicated. We shall not deal with this aspect of the problem. Instead, we refer the reader to the excellent memoir by Nörlund (1955).

For further discussion on the analytic continuation of the function $G_{p,p}^{m,n}(z)$, see 5.7(10, 11) and 5.10(21–23).

5.4. Elementary Properties of the G-Function

As previously remarked, the G-function is an analytic function of z with a branch point at the origin. It is symmetric in the parameters $a_1, \ldots, a_n$; $a_{n+1}, \ldots, a_p$; $b_1, \ldots, b_m$; and $b_{m+1}, \ldots, b_q$. Thus, if one of the a_h's, $h = 1, 2, \ldots, n$, is equal to one of the b_j's, $j = m + 1, \ldots, q$, the G-function reduces to one of lower order. For example,

$$G_{p,q}^{m,n}\left(z \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{q-1}, a_1 \end{matrix}\right) = G_{p-1,q-1}^{m,n-1}\left(z \middle| \begin{matrix} a_2, \ldots, a_p \\ b_1, \ldots, b_{q-1} \end{matrix}\right), \qquad n, p, q \geqslant 1. \tag{1}$$

Similarly, if one of the a_h's, $h = n + 1, \ldots, p$, is equal to one of the b_j's, $j = 1, 2, \ldots, m$, then the G-function reduces to one of lower order For example,

$$G^{m\ n}_{p\ q}\left(z \left| \begin{matrix} a_1, & \ldots, a_{p-1}, & b_1 \\ b_1, & b_2, & \ldots, b_q \end{matrix}\right.\right) = G^{m-1\ n}_{p-1\ q-1}\left(z \left| \begin{matrix} a_1, & \ldots, a_{p-1} \\ b_2, & \ldots, b_q \end{matrix}\right.\right) \qquad m, p, q \geqslant 1 \tag{2}$$

The important relation

$$G^{m\ n}_{p\ q}\left(z \left| \begin{matrix} a_p \\ b_q \end{matrix}\right.\right) = G^{n\ m}_{q\ p}\left(z^{-1} \left| \begin{matrix} 1 - b_q \\ 1 - a_p \end{matrix}\right.\right), \qquad \arg\left(\frac{1}{z}\right) = -\arg z, \tag{3}$$

has already been noted Its significance lies in the fact that in the discussion of the G-function we can, without loss of generality, suppose that $p \leqslant q$ Another important result easily proved from 5 2(1) is that

$$z^{\sigma} G^{m\ n}_{p\ q}\left(z \left| \begin{matrix} a_p \\ b_q \end{matrix}\right.\right) = G^{m\ n}_{p\ q}\left(z \left| \begin{matrix} a_p + \sigma \\ b_q + \sigma \end{matrix}\right.\right) \tag{4}$$

If in the integrand of 5 2(1), we replace s by ks, k a positive integer, and use 2 3(1), then

$$G^{m\ n}_{p\ q}\left(z \left| \begin{matrix} a_p \\ b_q \end{matrix}\right.\right) = (2\pi)^u k^v G^{mk\ nk}_{pk\ qk}\left(\left\{\frac{z^k}{k^{k(q-p)}}\right\} \left| \begin{matrix} c_{1k}, c_{2k}, & \ldots, c_{pk} \\ d_{1k}, d_{2k}, & \ldots, d_{qk} \end{matrix}\right.\right),$$

$$u = (k-1)[\tfrac{1}{2}(p+q) - m - n] \qquad v = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \tfrac{1}{2}(p-q) + 1, \tag{5}$$

where c_{hk} and d_{hk} stand respectively for the set of parameters

$$\frac{a_h}{k}, \quad \frac{a_h+1}{k}, \quad \ldots, \quad \frac{a_h+k-1}{k}, \qquad h = 1, 2, \ldots, p,$$

$$\frac{b_h}{k}, \quad \frac{b_h+1}{k}, \quad \ldots, \quad \frac{b_h+k-1}{k}, \qquad h = 1, 2, \ldots, q$$

The following identities are readily proved from 5 2(1)

$$G^{m+1\ k}_{p+1\ q+1}\left(z \left| \begin{matrix} a, & a_p \\ a, & b_q \end{matrix}\right.\right) = (-)^r G^{m\ k}_{p\ q}\left(z \left| \begin{matrix} a_p \\ b_q \end{matrix}\right.\right), \qquad a_k = a + r, \quad r \text{ an integer} \tag{6}$$

$$G^{m\ n+1}_{p+1\ q+1}\left(z \left| \begin{matrix} a, & a_p \\ b_q, & b \end{matrix}\right.\right) = (-)^r G^{m+1\ n}_{p+1\ q+1}\left(z \left| \begin{matrix} a_p, & a \\ b, & b_q \end{matrix}\right.\right) \qquad q \geqslant m, \quad a - b = r,$$

r an integer or zero (7)

$$(1-a_1+b_1)\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)=G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1-1,\,a_2,\ldots,a_p\\b_q\end{matrix}\right)$$
$$+\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_1+1,\,b_2,\ldots,b_q\end{matrix}\right),$$
$$m,n\geqslant 1. \qquad (8)$$

$$(a_h-a_1)\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)=G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1-1,\,a_2,\ldots,a_p\\b_q\end{matrix}\right)$$
$$+\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_p-1\\b_q\end{matrix}\right),$$
$$1\leqslant n\leqslant p-1,\qquad h=p. \qquad (9)$$

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)=(2\pi i)^{-1}\left\{\exp(i\pi b_{m+1})\,G^{m+1,n}_{p,q}\left(ze^{-i\pi}\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)\right.$$
$$\left.-\exp(-i\pi b_{m+1})\,G^{m+1,n}_{p,q}\left(ze^{i\pi}\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)\right\},$$
$$m\leqslant q-1. \qquad (10)$$

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)=(2\pi i)^{-1}\left\{\exp(i\pi a_{n+1})\,G^{m,n+1}_{p,q}\left(ze^{-i\pi}\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)\right.$$
$$\left.-\exp(-i\pi a_{n+1})\,G^{m,n+1}_{p,q}\left(ze^{i\pi}\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)\right\},$$
$$n\leqslant p-1. \qquad (11)$$

$$\frac{d}{dz}\left\{z^{-b_1}\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)\right\}=-\,z^{-1-b_1}\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\1+b_1,\,b_2,\ldots,b_q\end{matrix}\right). \qquad (12)$$

$$\frac{d}{dz}\left\{z^{-b_h}\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)\right\}=z^{-1-b_h}G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_1,\ldots,b_{q-1},\,1+b_q\end{matrix}\right)$$
$$m<q,\quad h=q. \qquad (13)$$

$$\frac{d}{dz}\left\{z^{1-a_1}\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)\right\}=z^{-a_1}\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1-1,\,a_2,\ldots,a_p\\b_q\end{matrix}\right),\qquad n\geqslant 1. \qquad (14)$$

$$\frac{d}{dz}\left\{z^{1-a_h}\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)\right\}=-z^{-a_h}\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_p-1\\b_q\end{matrix}\right),$$
$$n<p,\quad h=p. \qquad (15)$$

The latter four expressions can be put into alternative forms by actually performing the differentiations on the left. Thus, for example, (14) is equivalent to

$$z\,\frac{d}{dz}\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)=G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1-1,\,a_2,\ldots,a_p\\b_q\end{matrix}\right)+(a_1-1)\,G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right),$$
$$n\geqslant 1. \qquad (16)$$

We also have

$$z^k \frac{d^k}{dz^k}\left\{G^{m\ n}_{p\ q}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right)\right\} = G^{m\ n+1}_{p+1\ q+1}\left(z \,\middle|\, \begin{matrix} 0,\ a_p \\ b_q,\ k \end{matrix}\right), \tag{17}$$

$$z^k \frac{d^k}{dz^k}\left\{G^{m\ n}_{p\ q}\left(z^{-1} \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right)\right\} = (-)^k\, G^{m\ n+1}_{p+1\ q+1}\left(z^{-1} \,\middle|\, \begin{matrix} 1-k,\ a_p \\ b_q,\ 1 \end{matrix}\right) \tag{18}$$

In most of the above relations, we must omit the branch point at $z = 0$ This can be relaxed for special values of the parameters Thus (12) is valid for $z = 0$ if $m = 1$ and b_1 is a positive integer

Note that many known relations for the ${}_2F_1$, ${}_1F_1$, and more generally the ${}_pF_q$, can be readily deduced from the formulas of this section in view of the connecting formulas in 5 2(12–17)

5.5 Multiplication Theorems

Throughout this section $z \neq 0$ and m, n, p, and q are integers with

$$q \geqslant 1, \qquad 0 \leqslant n \leqslant p \leqslant q, \qquad 0 \leqslant m \leqslant q$$

Then we prove that

$$G^{m\ n}_{p\ q}\left(zw \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = \sum_{k=0}^{\infty} \frac{(w-1)^k}{k!}\, G^{m\ n+1}_{p+1\ q+1}\left(z \,\middle|\, \begin{matrix} 0,\ a_p \\ b_q,\ k \end{matrix}\right), \tag{1}$$

valid under the five cases enumerated below

CASE 1 $p < q, |w-1| < 1$

CASE 2 $p = q, (-)^{m+n-p}z \neq 1, |w-1| < 1$ if $(-)^{m+n-p}R(z) \leqslant \frac{1}{2}$,

$$|w-1| < \left|1 - \frac{(-)^{m+n-p}}{z}\right| \quad \text{if} \quad (-)^{m+n-p}R(z) \geqslant \tfrac{1}{2}$$

CASE 3 $p = q, \quad m+n-p \geqslant 1, \quad |\arg z| \leqslant (m+n-p-\frac{1}{2})\pi$, $|w-1| < 1$

CASE 4 $m = 1, p < q, b_1$ is a nonnegative integer That is, under these restrictions (1) is valid for all z and w

CASE 5 $m = 1, p = q, b_1$ is a nonnegative integer,

$$(-)^{1+n-p}z \neq 1, \qquad |w-1| < \left|1 - \frac{(-)^{1+n-p}}{z}\right|$$

REMARKS. In each case, the values of the multivalued functions $G^{m,n}_{p,q}(zw)$ and $G^{m,n+1}_{p+1,q+1}(z)$ are connected in the following way. For a chosen value of $\arg z$, the value of $\arg zw$ is determined in a prescribed fashion so that the values of the respective G-functions are uniquely determined by means of the integral representation 5.2(1). Thus for Case 1,

$$\arg zw = \arg z + \arg w, \qquad |\arg w| < \pi/2. \tag{2}$$

In Case 2, we make a cross cut in the z-plane along the straight line from $(-)^{m+n-p}$ to $(-)^{m+n-p} + \epsilon i\infty$ where $\epsilon = +1$ or -1 according as $I(z) \leqslant 0$ or $\geqslant 0$, respectively. If $I(z) = 0$, we may take $\epsilon = 1$ or $\epsilon = -1$ in which event $G^{m,n}_{p,p}(zw)$ and $G^{m,n}_{p,p}(z)$ may depend on the choice of ϵ. For a chosen value of $\arg z$, the values of $G^{m,n}_{p,p}(z/2|z|)$ and $G^{m,n+1}_{p+1,p+1}(z/2|z|)$ are uniquely determined by 5.2(1). The values of $G^{m,n}_{p,p}(z)$ and $G^{m,n+1}_{p+1,p+1}(z)$ are derived from those of $G^{m,n}_{p,p}(z/2|z|)$ and $G^{m,n+1}_{p+1,p+1}(z/2|z|)$, respectively, by analytic continuation in the cut plane along the straight line, suitably indented at $(-)^{m+n-p}$ if necessary, which joins $z/2|z|$ to z. The value of $G^{m,n}_{p,p}(zw)$ is obtained from that of $G^{m,n}_{p,p}(z)$ by analytic continuation in the cut plane along the straight line from z to zw.

In Case 3, the values of $G^{m,n}_{p,p}(zw)$ and $G^{m,n+1}_{p+1,p+1}(z)$ are connected as follows. For a chosen value of $\arg z$, $|\arg z| \leqslant (m + n - p - \frac{1}{2})\pi$, the value of $\arg(zw)$ follows from (2) and the values of $G^{m,n}_{p,p}(zw)$ and $G^{m,n+1}_{p+1,p+1}(z)$ follows from 5.2(1). The values of $G^{m,n}_{p,p}(zw)$ $\{G^{m,n+1}_{p+1,p+1}(z)\}$ are gotten from those of $G^{m,n}_{p,p}(zw/2|zw|)$ $\{G^{m,n+1}_{p-1,p+1}(zw/2|zw|)\}$ by analytic continuation along the straight line which joins $zw/2|zw|$ to zw $\{z/2|z|$ to $z\}$, respectively.

For Case 5, the plane is cut as in Case 2 with $m = 1$. In connection with this case, see also (8).

Note that Case 3 gives a better result than Case 2 when $m + n - p \geqslant 1$ and $|\arg z| \leqslant (m + n - p - \frac{1}{2})\pi$, for if, also, $(-)^{m+n-p}R(z) > \frac{1}{2}$, which is possible only when $m + n - p \geqslant 2$, the domain defined by $|w - 1| < 1$ includes that defined by $|w - 1| < |1 - [(-)^{m+n-p}/z]|$. Similarly, Case 5 gives a better result than Case 2 when $m = 1$ and b_1 is a positive integer or zero, for when $(-)^{1+n-p}R(z) \leqslant \frac{1}{2}$, the region defined by $|w - 1| < 1$ is included in the region defined by $|w - 1| < |1 - [(-)^{1+n-p}/z]|$.

PROOF. By Taylor's theorem

$$G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = \sum_{k=0}^{\infty} \frac{(x-\xi)^k}{k!} \frac{d^k}{d\xi^k} G^{m,n}_{p,q}\left(\xi \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right). \tag{3}$$

Replace x and ξ by zw and z, respectively, in (3) and use 5.4(17) to get (1) formally. To complete the proof, we need to determine the radius

of the largest circle in the x-plane with center at ξ such that the G-function on the left-hand side of (3) is analytic everywhere on and within this circle

It is convenient to first treat Cases 1 and 4 together If $p < q$ the only singularity of $G_{p\,q}^{m\,n}(x)$ is a branch point at the origin unless $m = 1$ and b_1 is a positive integer or zero, in which event $G_{p\,q}^{m\,n}(x)$ is analytic at the origin In the latter situation (3) is valid for all values of x and ξ, and this leads to Case 4 We now exclude the Case 4 hypothesis Let $\xi \neq 0$ be an arbitrary point in the x-plane The point ξ does not necessarily have its principal value Then $G_{p\,q}^{m\,n}(x)$ is analytic in the interior of a circle C with center at ξ and radius $|\xi|$ for every x in C provided that $\arg x$ is uniquely determined by some appropriate agreement Let us agree that

$$|\arg x - \arg \xi| < \pi/2$$

Then (3) is valid for $|x - \xi| < |\xi|$ and this proves Case 1

Next we turn to the proof of Cases 2 and 5 If $p = q$, then $G_{p\,p}^{m\,n}(x)$ has a singularity at $x = (-)^{m+n-p}$ and a branch point at the origin unless $m = 1$ and b_1 is a nonnegative integer, in which event $G_{p\,p}^{m\,n}(x)$ is analytic at the origin Let us cut the x plane and choose $\arg x$, etc, as in the above remarks associated with Case 2 (There put $z = x$ and $w = \xi$) Let $\xi \neq 0$, $\xi \neq (-)^{m+n-p}$ be an arbitrary point in the x plane If there is a branch point at the origin and ξ is not farther from the origin than it is to $(-)^{m+n-p}$, that is $|\xi| \leqslant |\xi - (-)^{m+n-p}|$, which implies that $(-)^{m+n-p}R(\xi) \leqslant \frac{1}{2}$, then (3) is valid for $|x - \xi| < |\xi|$ and we get the first part of the statement in Case 2 Suppose now that $(-)^{m+n-p}R(\xi) \geqslant \frac{1}{2}$ Then (3) holds for $|x - \xi| < |(-)^{m+n-p} - \xi|$ and we have the second part of Case 2 and Case 5

For Case 3, in view of the above remarks, with $m + n - p \geqslant 1$ $G_{p\,p}^{m\,n}(x)$ is analytic in the entire sector $x \neq 0$, $|\arg x| < (m + n - p)\pi$ When $m + n - p \geqslant 1$, $\xi \neq 0$, and $|\arg \xi| \leqslant (m + n - p - \frac{1}{2})\pi$, the interior of a circle with center at ξ and radius $|\xi|$ lies within this sector so that $G_{p\,p}^{m\,n}(x)$ is analytic in the interior of this circle and Case 3 follows This completes the proof of (1)

Equation (1) was first given by Meijer (1941c) See also Meijer (1952 1956, 1952, p 376), Knottnerus (1960) It is a generalization of well known results for Bessel functions, confluent hypergeometric functions, and hypergeometric functions

It is clear from the discussion surrounding Cases 2 and 5 of (1) that the conditions of validity depend on the manner in which the complex z-plane is cut to insure that the G function is single valued It is of interest to take a closer look at this situation for a particular case of (1)

Let $m = 1$, $n = p$, and replace q by $q + 1$. Let $b_1 = \mu$, $1 + b_1 - a_j = \alpha_j$, $j = 1, 2, \ldots, p$, and $1 + b_1 - b_{h+1} = \rho_h$, $h = 1, 2, \ldots, q$. Finally, replace z by $-z$. Then (1) becomes

$$w^{\mu}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\, zw\right) = \sum_{k=0}^{\infty} \frac{(1-w)^k(-\mu)_k}{k!}\,{}_{p+1}F_{q+1}\left(\begin{matrix}\mu+1, \alpha_p\\ \mu+1-k, \rho_q\end{matrix}\middle|\, z\right). \tag{4}$$

The conditions of validity are:

$$\begin{aligned}&p \leqslant q,\ |w-1| < 1 \text{ unless } \mu \text{ is a positive integer or zero, in which case the expansion is valid for all } z \text{ and } w;\ p = q+1,\ z \neq 1,\\ &|w-1| < 1 \text{ if } R(z) \leqslant \tfrac{1}{2},\ |w-1| < |1-1/z| \text{ if } R(z) \geqslant \tfrac{1}{2},\\ &\text{unless } \mu \text{ is a positive integer or zero, in which event the expansion is valid for } |w-1| < |1-1/z|.\end{aligned} \tag{5}$$

Here, when $p = q + 1$, the z-plane is cut along the straight line from 1 to $1 + \epsilon i\infty$ where $\epsilon = +1$ or -1 according as $I(z) \leqslant 0$ or $\geqslant 0$, respectively. For the ${}_{p+1}F_p(z)$ function, when $p = 1$, our usual practice has been to cut the plane along the real axis from 1 to $+\infty$ [see 3.6(1, 6)]. If we follow this procedure for the development of (4), then for $p = q + 1$, the conditions of validity are:

$$\begin{aligned}&z \neq 1,\ |w-1| < 1 \text{ if } R(z) \leqslant \tfrac{1}{2},\ |w-1| < |1-1/z| \text{ if } \tfrac{1}{2} \leqslant R(z) \leqslant 1,\ |w-1| < |I(z)|/|z| \text{ if } R(z) \geqslant 1, \text{ unless } \mu \text{ is a}\\ &\text{positive integer or zero, in which event (4) is valid for } z \neq 1,\\ &|w-1| < |1-1/z| \text{ if } R(z) \leqslant 1,\ |w-1| < |I(z)|/|z| \text{ if } R(z) \geqslant 1.\end{aligned} \tag{6}$$

Let us now reconsider (1) in the general situation when $p = q$. Suppose we cut the z-plane along the real axis from $(-)^{m+n-p}$ to $(-)^{m+n-p}\,\infty$. Then for a chosen value of $\arg z$, the value of $G^{m,n}_{p,p}(zw)$ is obtained from that of $G^{m,n}_{p,p}(z)$ by analytic continuation along any path connecting z and zw which does not intersect the cross cut. Let ξ be an arbitrary point in the x-plane, $\xi \neq 0$, $\xi \neq (-)^{m+n-p}$. If $(-)^{m+n-p}R(\xi) \leqslant 1$, then (3) is valid for $|x - \xi| < M$, $M = \min(|\xi|, |\xi - (-)^{m-n-p}|)$. If $(-)^{m+n-p}R(\xi) \geqslant 1$, then (3) is valid for $|x - \xi| < |I(\xi)|$. With x and ξ replaced by zw and z, respectively, and the complex plane cut as above, we have that (1) is valid for *Cases* 2* and 5* as follows.

$$\begin{aligned}&\textsc{Case 2*}.\ p = q,\ z \neq (-)^{m+n-p},\ |w-1| < 1 \text{ if } (-)^{m+n-p}R(z) \leqslant \tfrac{1}{2};\\ &|w-1| < |1-[(-)^{m+n-p}/z]| \text{ if } \tfrac{1}{2} \leqslant (-)^{m+n-p}R(z) \leqslant 1;\ |w-1| < |I(z)|/|z| \\ &\text{if } (-)^{m+n-p}R(z) \geqslant 1.\end{aligned} \tag{7}$$

CASE 5* $m = 1, p = q$ b_1 is a nonnegative integer,

$$(-)^{m+n-p}z \neq 1, \qquad |w-1| < \left|1 - \frac{(-)^{1+n-p}}{z}\right| \quad \text{if} \quad (-)^{1+n}\ {}^{p}R(z) \leqslant 1,$$

$$|w-1| < \frac{|I(z)|}{|z|} \quad \text{if} \quad (-)^{1+n}\ {}^{p}R(z) \geqslant 1 \tag{8}$$

This concludes our discussion of (1)

With $m, n, p,$ and q as in (1), then

$$G^{m\ n}_{p\ q}\left(zw \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) = \sum_{k=0}^{\infty} \frac{(1-1/w)^k}{k!} G^{m\ n+1}_{p+1\ q+1}\left(z \middle| \begin{matrix} 1-k, a_p \\ b_q, 1 \end{matrix}\right), \qquad z \neq 0, \tag{9}$$

is valid under the following three cases

CASE 1 $p < q$ $R(w) > \frac{1}{2}$

CASE 2 $p = q, (-)^{m+n}\ {}^{p}z \neq 1, R(w) > \frac{1}{2}$ if $|1 - (-)^{m+n}\ {}^{p}z| \geqslant 1,$

$$|1/w - 1| < |1 - (-)^{m+n}\ {}^{p}z| \quad \text{if} \quad |1 - (-)^{m+n-p}z| \leqslant 1 \tag{10}$$

CASE 3 $p = q$ $m + n - p \geqslant 1,$ $|\arg z| \leqslant (m + n - p - \frac{1}{2})\pi,$ $R(w) > \frac{1}{2}.$

We omit proof

Some other useful expansions can be derived from (1) and (9) We first consider (1) with $m < q$ If $b_h = 0$ for $h = q$, in view of the symmetric properties of the G-function [see 5 4(1)], we have

$$G^{m\ n}_{p\ q}\left(zw \middle| \begin{matrix} a_p \\ b_{q-1}, 0 \end{matrix}\right) = \sum_{k=0}^{\infty} \frac{(w-1)^k}{k!} G^{m\ n}_{p\ q}\left(z \middle| \begin{matrix} a_p \\ b_{q-1}, k \end{matrix}\right) \tag{11}$$

Multiply the latter by z^{b} and use 5 4(4) With an obvious change of notation, we get

$$G^{m\ n}_{p\ q}\left(zw \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) = w^{b_h} \sum_{k=0}^{\infty} \frac{(w-1)^k}{k!} G^{m\ n}_{p\ q}\left(z \middle| \begin{matrix} a_p \\ b_{q-1}, k + b_q \end{matrix}\right), \qquad h = q \tag{12}$$

From 5 4(7), we see that (1) can be replaced by

$$G^{m\ n}_{p\ q}\left(zw \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) = \sum_{k=0}^{\infty} \frac{(1-w)^k}{k!} G^{m+1\ n}_{p+1\ q+1}\left(z \middle| \begin{matrix} a_p, 0 \\ k, b_q \end{matrix}\right) \tag{13}$$

and if $b_1 = 0$, we get

$$G^{m\ n}_{p\ q}\left(zw \middle| \begin{matrix} a_p \\ 0, b_2, \ldots, b_q \end{matrix}\right) = \sum_{k=0}^{\infty} \frac{(1-w)^k}{k!} G^{m\ n}_{p\ q}\left(z \middle| \begin{matrix} a_p \\ k, b_2, \ldots, b_q \end{matrix}\right) \tag{14}$$

The latter is also valid for $m = 0$ since both sides vanish. Multiply both sides of (14) by z^β, use 5.4(4), change notation, and so find

$$G_{p,q}^{m,n}\left(zw \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = w^{b_1} \sum_{k=0}^{\infty} \frac{(1-w)^k}{k!} G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_p \\ k+b_1, b_2, \ldots, b_q \end{matrix}\right). \tag{15}$$

Similarly, from (9), if $n \geqslant 1$, we have

$$G_{p,q}^{m,n}\left(zw \,\middle|\, \begin{matrix} 1, a_2, \ldots, a_p \\ b_q \end{matrix}\right) = \sum_{k=0}^{\infty} \frac{(1-1/w)^k}{k!} G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} 1-k, a_2, \ldots, a_p \\ b_q \end{matrix}\right), \tag{16}$$

$$G_{p,q}^{m,n}\left(zw \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = w^{a_1-1} \sum_{k=0}^{\infty} \frac{(1-1/w)^k}{k!} G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1-k, a_2, \ldots, a_p \\ b_q \end{matrix}\right). \tag{17}$$

Finally, if $n < p$, (9) can be replaced by

$$G_{p,q}^{m,n}\left(zw \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = w^{a_h-1} \sum_{k=0}^{\infty} \frac{(1/w-1)^k}{k!} G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_{p-1}, a_p-k \\ b_q \end{matrix}\right), \quad h = p. \tag{18}$$

For generalizations of the expansions of this section, see 9.2(3, 5).

5.6. Integrals Involving G-Functions

5.6.1. $\int_0^\infty y^{s-1} G_{p,q}^{m,n}\left(\eta y \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dy$

From the definition of the G-function, 5.2(1), and the Mellin inversion formula [see Titchmarch (1948)] we have

$$\int_0^\infty y^{s-1} G_{p,q}^{m,n}\left(\eta y \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dy = \frac{\eta^{-s} \prod_{j=1}^{m} \Gamma(b_j+s) \prod_{j=1}^{n} \Gamma(1-a_j-s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s) \prod_{j=n+1}^{p} \Gamma(a_j+s)}, \tag{1}$$

valid under the seven cases enumerated below. We put

$$\delta = m + n - \tfrac{1}{2}(p+q), \qquad \xi = m + n - p. \tag{2}$$

We have need for the following conditions:

$$-\min_{1 \leqslant h \leqslant m} R(b_h) < R(s) < 1 - \max_{1 \leqslant j \leqslant n} R(a_j), \tag{3}$$

$$-\min_{1 \leqslant h \leqslant m} R(b_h) < R(s) < 1 - \max_{1 \leqslant j \leqslant p} R(a_j), \tag{3*}$$

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right\} + \tfrac{1}{2}(q+1-p) > (q-p)\, R(s). \tag{4}$$

It is understood that if $n = 0$, that part of (3) involving $1 - \max R(a_j)$ is treated as empty Thus (3) becomes $-\min R(b_h) < R(s)$, $1 \leqslant h \leqslant m$ Similar conventions are employed throughout

The given conditions in the cases for the validity of (1) in the main stem from the requirements that the integrand of (1) has proper behavior near the origin and infinity to insure that the integral is convergent We omit details for the derivation of the various cases, but we present a sketch of how they are obtained The behavior of the G-function near the origin is governed by 5 2(7) and if the G-function near infinity is algebraic as in Theorems 1, 6 1, and 7 of 5 10, then the integral in (1) converges provided that (3) holds This situation is covered by Case 1 When $n = 0$ and $\arg \eta$ is suitably restricted, the G-function for large argument is an exponential decay (see Theorem 2 of 5 10) and this situation is covered by Case 2 When the G-function near infinity is algebraic and sinusoidal, we get Case 3 Here the descriptive properties of the G-function can be deduced from 5 10(2 5, 9, 12, 13) In these situations condition (4) arises, since for $a > 0$,

$$\left| \int_a^\infty x^\alpha e^{i\beta x^\gamma} dx \right| < \infty \quad \text{if} \quad \beta > 0, \quad 0 < \gamma \leqslant 1, \quad R(\alpha) < \gamma - 1 \tag{5}$$

Note that the latter integral is finite when $\beta = 0$ if $R(\alpha) < -1$ When $q = p + 2$, $q = p + 1$, or $q = p$, the restrictions on $\arg \eta$ can be changed provided we also alter the condition (3) to (3*) The behavior of the G-function near infinity in these situations can be deduced from 5 10(16, 17, 19, 20, 22, 23) and conditions for the validity of (1) are given by the additional Cases 4–7

In all the following cases, $p \leqslant q$ If $p \geqslant q$, then

$$\begin{aligned} \int_0^\infty y^{s-1} G^{m\ n}_{p\ q}\left(\eta y \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) dy &= \int_0^\infty y^{s-1} G^{n\ m}_{q\ p}\left(\frac{1}{\eta y} \middle| \begin{matrix} 1 - b_q \\ 1 - a_p \end{matrix}\right) dy \\ &= \int_0^\infty x^{-s-1} G^{n\ m}_{q\ p}\left(\frac{x}{\eta} \middle| \begin{matrix} 1 - b_q \\ 1 - a_p \end{matrix}\right) dx \end{aligned} \tag{6}$$

Thus conditions for the validity of (1) when $p \geqslant q$ follow from the cases for $p \leqslant q$ by an appropriate change of notation, that is, $p \leftrightarrow q$, $m \leftrightarrow n$, replace ξ, a_h, b_h, s, and $\arg \eta$ by $m + n - q$, $1 - b_j$, $1 - a_h$, $-s$, and $-\arg \eta$, respectively

The cases for the validity of (1) are as follows

CASE 1 $1 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, (3), $\delta > 0$, $\eta \neq 0$, $|\arg \eta| < \delta\pi$ Equation (1) is also valid under the conditions (3), $\delta > 0$, $\eta \neq 0$, $|\arg \eta| < \delta\pi$ for $p \geqslant 1$, $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q = p + 1$ (but exclude $n = 0$ and $m = p + 1$), or $p \geqslant 1$, $0 \leqslant n \leqslant p$, $0 \leqslant m \leqslant q = p$

provided in this last instance we exclude $|\arg\eta| = (\delta - 2k)\pi$, $k = 0, 1, \ldots, [\delta/2]$.

CASE 2. $n = 0$, $1 \leqslant p + 1 \leqslant m \leqslant q$, (3), $\delta > 0$, $|\arg\eta| < \delta\pi$.

CASE 3. $0 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, (3), (4), $\delta > 0$, $|\arg\eta| = \delta\pi$ or $0 \leqslant n \leqslant p \leqslant q - 2$, (3), (4), $\delta = 0$, $\arg\eta = 0$.

ADDITIONAL CASES. If $q = p + 2$, $q = p + 1$, or if $q = p$, the restrictions on $\arg\eta$ in the above cases can be altered provided that in (3) the inequality involving $R(a_j)$ holds for $1 \leqslant j \leqslant p$. Thus condition (3) becomes (3*) as noted. In the sequel, λ is an integer (positive or negative) or zero.

CASE 4. $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q = p + 2$, (3*), (4), $\eta \neq 0$, $\arg\eta = (\delta + 2\lambda)\pi$, λ is arbitrary if $\delta \leqslant 0$, but if $\delta \geqslant 1$, $\lambda \geqslant 0$ or $\lambda \leqslant -(1 + \delta)$. However, if $\delta \geqslant 0$, and $\lambda = 0$, then replace (3*) by (3).

CASE 5. $p \geqslant 1$, $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q = p + 1$, (3*), $\eta \neq 0$, $(\delta + 2\lambda - 1)\pi < \arg\eta < (\delta + 2\lambda)\pi$, λ arbitrary, but exclude $n = \lambda = 0$ and $m = p + 1$ which is covered by Case 2. Also $(\delta + 2\lambda)\pi \leqslant \arg\eta < (\delta + 2\lambda + 1)\pi$ if $\xi \geqslant 2$ and $-\xi < \lambda < 0$ or $\arg\eta = (\delta + 2\lambda - 1)\pi$ if $\xi \geqslant 2$, $1 - \xi < \lambda < 1$. However, if $n \geqslant 1$, $\xi \geqslant 1$, and $1 - \xi \leqslant \lambda \leqslant 0$ then replace (3*) by (3).

CASE 6. $p \geqslant 1$, $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q = p + 1$, (3*), (4), $\eta \neq 0$, $\arg\eta = (\delta + 2\lambda)\pi$, λ is arbitrary if $\xi \leqslant 1$, but if $\xi \geqslant 2$, $\lambda \geqslant 0$ or $\lambda \leqslant -\xi$. Also $\arg\eta = (\delta + 2\lambda - 1)\pi$, λ is arbitrary if $\xi \leqslant 1$, but if $\xi \geqslant 2$, $\lambda \geqslant 1$ or $\lambda \leqslant 1 - \xi$.

CASE 7. $p \geqslant 1$, $0 \leqslant n \leqslant p$, $0 \leqslant m \leqslant q = p$, (3*), $\eta \neq 0$, $(\delta + 2\lambda - 2)\pi < \arg\eta < (\delta + 2\lambda)\pi$, λ is arbitrary, but if $\delta \geqslant 1$ and $1 - \delta \leqslant \lambda \leqslant 0$, then replace (3*) by (3) and exclude $|\arg\eta| = (\delta - 2k)\pi$, $k = 0, 1, \ldots, [\delta/2]$.

5.6.2. Integral of the Product of Two G-Functions

We prove that

$$\int_0^\infty G^{m,n}_{p,q}\left(\eta x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) G^{\mu,\nu}_{\sigma,\tau}\left(\omega x \,\middle|\, \begin{matrix} c_\sigma \\ d_\tau \end{matrix}\right) dx$$

$$= \frac{1}{\eta} G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}\left(\frac{\omega}{\eta} \,\middle|\, \begin{matrix} -b_1, \ldots, -b_m, c_\sigma, -b_{m+1}, \ldots, -b_q \\ -a_1, \ldots, -a_n, d_\tau, -a_{n+1}, \ldots, -a_p \end{matrix}\right)$$

$$= \frac{1}{\omega} G^{m+\nu,n+\mu}_{p+\tau,q+\sigma}\left(\frac{\eta}{\omega} \,\middle|\, \begin{matrix} a_1, \ldots, a_n, -d_\tau, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, -c_\sigma, b_{m+1}, \ldots, b_q \end{matrix}\right), \tag{1}$$

is valid under nine cases enumerated below.

PROOF Replace $G^{\mu\,0}_{\sigma\,\tau}(\omega x)$ in the above integrand by its integral representation 5 2(1), interchange the order of integration, and evaluate the inner integral using 5 6 1(1) Then apply 5 2(1) to the integral which remains To reverse the order of integration, we require that the integrals involved be absolutely convergent However, once (1) is established, the conditions required for absolute convergence can be weakened on appeal to the theory of analytic continuation The same remarks apply to all theorems of 5 6 involving change of order of integration unless stated to the contrary In this connection, the conditions we give for validity of the integrals are most always weaker than those given by the authors we quote

Equation (1) is due to Meijer (1941a) who gives five cases for its validity Our conditions for validity described below seem to be the most thoroughly known

The given conditions arise in much the same way as those for 5 6 1(1) Again details of proof are omitted, but we sketch how they are found The behavior of the G-functions near the origin is governed by 5 2(7), which leads to the condition (3) Near infinity, we make use of the asymptotic behavior and analytic continuation of the G-function as presented in 5 10 From these considerations the condition (4), or (4) suitably modified, see later discussion, always arises In Case 1 the behavior of both G functions near infinity is algebraic but not sinusoidal, and the conditions arise in the same manner as for Case 1 of 5 6 1(1) When $G^{m\,n}_{p\,q}(\eta x)$ is algebraic but not sinusoidal as in Case 1, and $G^{\mu\,0}_{\sigma\,\tau}(\omega x)$ is algebraic and sinusoidal, we get Case 2 Case 3 arises when both G-functions are algebraic and sinusoidal

When $\nu = 0$ and $\arg \omega$ is suitably restricted, $G^{\mu\,0}_{\sigma\,\tau}(\omega x)$ is an exponential decay for $x \to +\infty$, see Theorem 2 of 5 10 In this event the integral in (1) with lower limit $a > 0$ is defined and converges for all meaningful values of m, n, etc , provided $G^{m\,n}_{p\,q}(\eta x)$ is not exponentially increasing as $x \to +\infty$, unless $q - p > \tau - \sigma$ This situation is covered by Case 4 The conditions of validity for (1) when $G^{\mu\,0}_{\sigma\,\tau}(\omega x)$ and $G^{m\,n}_{p\,q}(\eta x)$ are exponentially decreasing and increasing, respectively, for $x \to +\infty$ and for $q - p = \tau - \sigma$, are detailed in Case 5 The conditions under which the G-function is exponentially increasing for large argument can be deduced from 5 10(6–8, 10, 11, 14, 15, 18)

In Cases 1–4, the restrictions on $\arg \eta$ can be altered if $q = p + 2$, $q = p + 1$, or $q = p$ as in Cases 4–7 of 5 6 1(1), provided that in the present instance, condition (4) holds for $j = 1, 2, \ldots, p$ Likewise, if $\tau = \sigma + 1$ or $\tau = \sigma + 2$, the inequalities involving $\arg \omega$ can be changed as in Cases 4–7 of 5 6 1(1), if in that notation, $m, n, p, q, \delta,$

and ξ are replaced by $\mu, \nu, \sigma, \tau, \rho$, and ζ, respectively, provided that condition (4) holds for $h = 1, 2,..., \sigma$. Obviously the inequalities involving $\arg \eta$ and $\arg \omega$ can be altered simultaneously as noted if (4) is true for $j = 1, 2,..., p$, $h = 1, 2,..., \sigma$. We omit these details.

In Cases 1–5, $p \leqslant q$ and $\sigma \leqslant \tau$. Suppose $p \geqslant q$ and $\sigma \leqslant \tau$. Then the left-hand side of (1) may be expressed as

$$\int_0^\infty G_{q,p}^{n,m}\left(\frac{1}{\eta x}\,\middle|\,\begin{matrix}1-b_q\\1-a_p\end{matrix}\right) G_{\sigma,\tau}^{\mu,\nu}\left(\omega x\,\middle|\,\begin{matrix}c_\sigma\\d_\tau\end{matrix}\right) dx.$$

The conditions of validity for this situation are described by Cases 6–9. Here neither of the G-functions for large values of its argument can be of exponential growth. If $p \geqslant q$ and $\tau \geqslant \sigma$, then in view of 5.4(3, 4) the left-hand side of (1) can be expressed as

$$(\eta\omega)^{-1}\int_0^\infty G_{q,p}^{n,m}\left(\frac{x}{\eta}\,\middle|\,\begin{matrix}-b_q\\-a_p\end{matrix}\right) G_{\tau,\sigma}^{\nu,\mu}\left(\frac{x}{\omega}\,\middle|\,\begin{matrix}-d_\tau\\-c_\sigma\end{matrix}\right) dx,$$

so that with appropriate change of notation Cases 1–5 may be applied.

It is convenient to put

$$\begin{aligned}\delta &= m + n - \tfrac{1}{2}(p+q), & \rho &= \mu + \nu - \tfrac{1}{2}(\sigma+\tau),\\ \xi &= m + n - p, & \zeta &= \mu + \nu - \sigma.\end{aligned} \tag{2}$$

We have need for the following conditions:

$$R(b_j + d_h) > -1, \qquad j = 1, 2,..., m, \quad h = 1, 2,..., \mu. \tag{3}$$

$$R(a_j + c_h) < 1, \qquad j = 1, 2,..., n, \quad h = 1, 2,..., \nu. \tag{4}$$

$$a_j - b_h \text{ is not a positive integer}, \qquad = 1, 2,..., n, \quad h = 1, 2,..., m. \tag{5}$$

$$c_j - d_h \text{ is not a positive integer}, \qquad j = 1, 2,..., \nu, \quad h = 1, 2,..., \mu. \tag{6}$$

$$\eta \neq 0, \qquad \omega \neq 0. \tag{7}$$

$$R\left\{\sum_{h=1}^{\sigma} c_h - \sum_{h=1}^{\tau} d_h\right\} + \tfrac{1}{2}(\tau + 1 - \sigma) > (\tau - \sigma)\, R(a_j), \qquad j = 1, 2,..., n. \tag{8}$$

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right\} + \tfrac{1}{2}(q + 1 - p) > (q - p)\, R(c_j), \qquad j = 1, 2,..., \nu. \tag{9}$$

$$u = \frac{R(\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h)}{q-p} \qquad v = \frac{R(\sum_{h=1}^{\sigma} c_h - \sum_{h=1}^{\tau} d_h)}{\tau - \sigma},$$

$$a = \frac{\frac{1}{2}}{q-p}, \qquad b = \frac{\frac{1}{2}}{\tau - \sigma},$$

$$\begin{aligned} a + u - b + v &> 0 \quad \text{if} \quad a \geqslant b, \\ -a + u + b + v &> 0 \quad \text{if} \quad a \leqslant b \end{aligned} \tag{10}$$

It is understood that if either n or ν is zero, then condition (4) is treated as empty, and similarly for the other conditions

The cases for the validity of (1) follow

CASE 1 $1 \leqslant n \leqslant p < q, 1 \leqslant m \leqslant q, 1 \leqslant \nu \leqslant \sigma < \tau, 1 \leqslant \mu \leqslant \tau$, (3)–(7), $\delta > 0, |\arg \eta| < \delta\pi, \rho > 0, |\arg \omega| < \rho\pi$ Equation (1) is also valid under these same conditions if the inequalities involving m, n, p, and q are replaced by $p \geqslant 1, 0 \leqslant n \leqslant p, 1 \leqslant m \leqslant q = p + 1$ (but exclude $n = 0$ and $m = p + 1$), or $p \geqslant 1, 0 \leqslant n \leqslant p$, $0 \leqslant m \leqslant q = p$ provided in this last instance we exclude $|\arg \eta| = (\delta - 2k)\pi$, $k = 0, 1, \ldots, [\delta/2]$ Likewise (1) is valid if in the latter statement we interchange the roles of m, n, etc , with μ, ν, etc Further, (1) is valid if we simultaneously alter the inequalities m, n, etc , μ, ν, etc , as above

CASE 2 Let m, n, p, q, δ, and η be as in Case 1, $0 \leqslant \nu \leqslant \sigma < \tau$, $1 \leqslant \mu \leqslant \tau$, (3)–(8), $\rho > 0, |\arg \omega| = \rho\pi$ Under the same circumstances (1) is also valid for $0 \leqslant \nu \leqslant \sigma \leqslant \tau - 2, \rho = 0, \arg \omega = 0$ Likewise (1) is valid if the roles of m, n, etc , and μ, ν, etc , are interchanged, whence the conditions (3)–(8) are replaced by (3)–(7), (9)

CASE 3 $0 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, $\delta > 0$, $|\arg \eta| = \delta\pi$ (or $0 \leqslant n \leqslant p \leqslant q - 2, \delta = 0, \arg \eta = 0$), $0 \leqslant \nu \leqslant \sigma < \tau$, $1 \leqslant \mu \leqslant \tau$, $\rho > 0, |\arg \omega| = \rho\pi$ (or $0 \leqslant \nu \leqslant \sigma \leqslant \tau - 2, \rho = 0, \arg \omega = 0$), (3)–(10), unless $a = b$, $|\eta/\omega| = 1$ and $\arg \omega + \arg \eta = 0$ occur simultaneously in which event replace (10) by $u + v - 2a > 0$ Also, if $a = b$, we require $|\arg(1 - (-)^{\delta+\rho}(\eta/\omega))| < \pi$

CASE 4 Let m, n, p, q, δ, ξ, and $\arg \eta$ be as in any of the seven cases associated with 5 6 1(1) $\nu = 0, 1 \leqslant \sigma + 1 \leqslant \mu \leqslant \tau$, (3)–(7), $\rho > 0$, $|\arg \omega| < \rho\pi$ Equation (1) is also valid for all other possibilities of m, n, etc , not enumerated above, provided $q - p > \tau - \sigma$

The restrictions required for the validity of (1) for the Case 4 situation but with $q - p = \tau - \sigma$ are given by Case 5 below

First, we record some conditions which will be needed.

$$
\begin{aligned}
&\nu = 0, && \sigma + 1 \leqslant \mu \leqslant \tau, \\
&\alpha = \frac{\arg \omega + \pi(\tau - \mu)}{\tau - \sigma} && \text{if} \quad 0 < \arg \omega < \rho\pi, \\
&\alpha = \frac{\arg \omega - \pi(\tau - \mu)}{\tau - \sigma} && \text{if} \quad -\rho\pi < \arg \omega < 0, \\
&\alpha = \frac{\pi(\tau - \mu)}{\tau - \sigma} && \text{if} \quad \arg \omega = 0 \quad \text{provided} \quad \rho > 0.
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&0 \leqslant n \leqslant p \leqslant q - 2, \qquad m + n - p \geqslant 1, \qquad \delta \leqslant 0, \\
&\beta = \frac{\arg \eta + \pi(q - m - n)}{q - p} \qquad \text{if} \quad 0 < \arg \eta < (m + n + 1 - p)\pi, \\
&\beta = \frac{\arg \eta - \pi(q - m - n)}{q - p} \qquad \text{if} \quad -(m + n + 1 - p)\pi < \arg \eta < 0, \\
&\beta = \frac{\pi(q - m - n)}{q - p} \qquad \text{if} \quad \arg \eta = 0 \quad \text{provided} \quad \delta < 0.
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&0 \leqslant n \leqslant p < q, \qquad 1 \leqslant m \leqslant q, \qquad \delta > 0, \\
&\beta = \frac{\arg \eta + \pi(q - m - n)}{q - p} \qquad \text{if} \quad \delta\pi < \arg \eta < (m + n - p + \epsilon)\pi, \\
&\beta = \frac{\arg \eta - \pi(q - m - n)}{q - p} \qquad \text{if} \quad -(m + n - p + \epsilon)\pi < \arg \eta < -\delta\pi, \\
&\epsilon = \tfrac{1}{2} \quad \text{if} \quad q = p + 1, \qquad \epsilon = 1 \quad \text{if} \quad q > p + 1.
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&0 \leqslant n \leqslant p \leqslant q - 2, \qquad 1 \leqslant m \leqslant q, \\
&\beta = \frac{\arg \eta + \pi(q - m - n - 2\lambda)}{q - p} \\
&\quad \text{if} \quad (m + n - p + 2\lambda - 1)\pi < \arg \eta < (m + n - p + 2\lambda + 1)\pi, \\
&\beta = \frac{(q - p - 1)\pi}{q - p} \\
&\quad \text{if} \quad \arg \eta = (m + n - p + 2\lambda - 1)\pi \quad \text{provided} \quad p < q - 2.
\end{aligned}
\tag{14}
$$

In (14), λ is an arbitrary integer if $m + n - p \leqslant 1$. If $m + n - p \geqslant 2$, λ is either an arbitrary integer $\geqslant 0$ or $\leqslant -(m + n - p)$.

$$
\begin{aligned}
&p \geqslant 1, \qquad 0 \leqslant n \leqslant p, \qquad 1 \leqslant m \leqslant q = p + 1, \\
&\beta = \frac{\arg \eta - \pi(m + n - p + 2\lambda - 1)}{q - p} \\
&\quad \text{if} \quad (m + n - p + 2\lambda - \tfrac{1}{2})\pi < \arg \eta < (m + n - p + 2\lambda + \tfrac{1}{2})\pi.
\end{aligned}
\tag{15}
$$

In (15) λ is an arbitrary integer $\geqslant 0$ or $\leqslant -(m+n-p)$ if $m+n-p \geqslant 2$ If $m+n-p \leqslant 1$, λ is an arbitrary integer

CASE 5 Under the conditions $q-p=\tau-\sigma$, (3)–(7), and either of the combinations (11), (12), (11), (13), (11), (14), (11), (15), (1) is valid when $\cos\alpha + |\eta\omega|^{1/(q-p)}\cos\beta > 0$ or when $\cos\alpha + |\eta\omega|^{1/(q-p)} \times \cos\beta = 0$ and $u+v-2a>0$ where u, v, and a are defined as in (10), provided further that $\arg(1-(-)^{q+p}(\eta\omega)) < \pi$

Next we present Cases 6–9 which give conditions for the validity of (1) when $p \geqslant q$ We have need for the condition

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right\} + (p-q)R(d_j) + \tfrac{1}{2}(p+1-q) > 0, \quad j=1,2,\ \ldots,\mu. \tag{16}$$

CASE 6 $1 \leqslant m \leqslant q < p$, $1 \leqslant n \leqslant p$, $1 \leqslant \nu \leqslant \sigma < \tau$, $1 \leqslant \mu \leqslant \tau$, (3)–(7), $\delta > 0$, $|\arg\eta| < \delta\pi$, $\rho > 0$, $|\arg\eta| < \delta\tau$ Equation (1) is also valid under these same conditions but with $q \geqslant 1$, $0 \leqslant m \leqslant q$, $1 \leqslant n \leqslant p = q+1$ or $q \geqslant 1$, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p = q$, provided in this last instance we exclude $|\arg\eta| = (\delta - 2k)\pi$, $k = 0, 1, \ldots, [\delta/2]$ Further, (1) is valid under the above conditions if the quantities μ, ν, σ, and τ are replaced as outlined in Case 1 Also (1) is valid for all m, n, etc., above and $\nu = 0$, $1 \leqslant \sigma+1 \leqslant \mu = \tau$, $\rho > 0$, $|\arg\omega| < \rho\pi$, and for all μ, ν, etc., above and $m = 0$, $1 \leqslant q+1 \leqslant n \leqslant p$, $\delta > 0$, $|\arg\eta| < \delta\tau$, and for m, n, etc., μ, ν, etc., as just related

CASE 7 Let m, n, p, q, δ, and $\arg\eta$ be as in Case 6, (3)–(8), $0 \leqslant \nu \leqslant \sigma < \tau$, $1 \leqslant \mu \leqslant \tau$, $\rho > 0$, $|\arg\omega| = \rho\pi$ or $0 \leqslant \nu \leqslant \sigma \leqslant \tau - 2$, $\rho = 0$, $\arg\omega = 0$

CASE 8 Let $\mu, \nu, \sigma, \tau, \rho$, and $\arg\omega$ be as in Case 1, (3)–(7), (16), $0 \leqslant m \leqslant q < p$, $1 \leqslant n \leqslant p$, $\delta > 0$, $|\arg\eta| = \delta\pi$ or $0 \leqslant m \leqslant q < p-2$, $\delta = 0$, $\arg\eta = 0$

CASE 9 m, n, etc., as in Case 8, μ, ν, etc., as in Case 7, (3)–(8), (16)

By specializing the parameters in (1) and employing the tables in Chapter VI we get many important integrals involving the functions of mathematical physics For example, with $\nu = \sigma$ and $\mu = 1$, in view of 5 2(12), we find

$$\int_0^\infty y^{d_1}\,{}_\sigma F_{\tau-1}\left(\begin{matrix}1+d_1-c_\sigma\\ 1+d_1-d_\tau^*\end{matrix}\middle|\,-\omega y\right) G_{p\,q}^{m\,n}\left(zy\middle|\begin{matrix}a_p\\ b_q\end{matrix}\right) dy$$

$$= \frac{\Gamma(1+d_1-d_1)}{\Gamma(1+d_1-c_1)}\,\omega^{-d_1-1}\,G_{p+\tau\,q+\sigma}^{m+\tau\,n+1}\left(\frac{z}{\omega}\middle|\begin{matrix}a_1, \ldots, a_n, -d_1, a_{n+1}, \ldots, a_p\\ b_1, \ldots, b_m, -c_\sigma, b_{m+1}, \ldots, b_q\end{matrix}\right) \tag{17}$$

valid under the conditions given for (1) (with $\mu = 1$ and $\nu = \sigma$) which we do not produce. In (17), put $\sigma = \tau = 1$. With an apparent change of notation, we get

$$\int_0^\infty y^{\alpha-1}(y+\beta)^{-\sigma}\, G^{m,n}_{p,q}\left(zy \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dy = \frac{\beta^{\alpha-\sigma}}{\Gamma(\sigma)}\, G^{m+1,n+1}_{p+1,q+1}\left(\beta z \,\middle|\, \begin{matrix} 1-\alpha, a_p \\ \sigma-\alpha, b_q \end{matrix}\right). \tag{18}$$

The cases of validity for (18) can be derived from Cases 1, 4 (there interchange m, n., etc., with μ, ν, etc.), 6, and 8, with $\mu = \nu = \sigma = \tau = 1$, $d_1 = \alpha - 1$, $c_1 = \alpha - \sigma$, and $\omega = 1/\beta$.

The Hankel-, Y-, K-, and $\mathbf{H}$ transforms are

$$\int_0^\infty x^{-a} J_\nu(2[\omega x]^{1/2})\, G^{m,n}_{p,q}\left(\eta x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dx$$

$$= \omega^{a-1}\, G^{m,n+1}_{p+2,q}\left(\frac{\eta}{\omega} \,\middle|\, \begin{matrix} a-\frac{1}{2}\nu, a_p, a+\frac{1}{2}\nu \\ b_q \end{matrix}\right), \tag{19}$$

$$\int_0^\infty x^{-a} Y_\nu(2[\omega x]^{1/2})\, G^{m,n}_{p,q}\left(\eta x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dx$$

$$= \omega^{a-1}\, G^{m,n+2}_{p+3,q+1}\left(\frac{\eta}{\omega} \,\middle|\, \begin{matrix} a+\frac{1}{2}\nu, a-\frac{1}{2}\nu, a_p, a+\frac{1}{2}\nu+\frac{1}{2} \\ b_q, a+\frac{1}{2}\nu+\frac{1}{2} \end{matrix}\right), \tag{20}$$

$$\int_0^\infty x^{-a} K_\nu(2[\omega x]^{1/2})\, G^{m,n}_{p,q}\left(\eta x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dx$$

$$= \frac{\omega^{a-1}}{2}\, G^{m,n+2}_{p+2,q}\left(\frac{\eta}{\omega} \,\middle|\, \begin{matrix} a+\frac{1}{2}\nu, a-\frac{1}{2}\nu, a_p \\ b_q \end{matrix}\right), \tag{21}$$

$$\int_0^\infty x^{-a} \mathbf{H}_\nu(2[\omega x]^{1/2})\, G^{m,n}_{p,q}\left(\eta x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dx$$

$$= \omega^{a-1}\, G^{m+1,n+1}_{p+3,q+1}\left(\frac{\eta}{\omega} \,\middle|\, \begin{matrix} a-\frac{1}{2}-\frac{1}{2}\nu, a_p, a-\frac{1}{2}\nu, a+\frac{1}{2}\nu \\ a-\frac{1}{2}-\frac{1}{2}\nu, b_q \end{matrix}\right), \tag{22}$$

respectively. Again we omit conditions of validity as these are readily deduced from the cases associated with (1). For numerous examples of (1), see Meijer (1936, 1941a).

For a final transform, we have

$$\int_0^\infty (z+y)^{-1}\, G^{m,n}_{p,q}\left(\eta y^2 \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dy = (2\pi)^{-1}\, G^{m+2,n+2}_{p+2,q+2}\left(\eta z^2 \,\middle|\, \begin{matrix} 0, \frac{1}{2}, a_p \\ 0, \frac{1}{2}, b_q \end{matrix}\right),$$

$$|\arg z| < \pi, \tag{23}$$

valid under the three cases cited below We have need for the conditions

$$R(b_h) > -\tfrac{1}{2} \quad \text{or} \quad R(b_h) > 0$$

according as $z \neq 0$ or $z = 0$, respectively,

$$h = 1, 2, \ldots, m \qquad R(a_j) < 1, \qquad j = 1, 2, \ldots, n,$$

$$a_j - b_h \text{ is not a positive integer, } j \text{ and } h \text{ as above} \tag{24}$$

$$R\left(\sum_{h=1}^{q} b_h - \sum_{h=1}^{p} a_h\right) < \tfrac{1}{2}(q + 1 - p) \tag{25}$$

Let m, n, etc , be as in Cases 1, 2, and 3 of 5 6 1(1) Then (23) is valid as for the latter cases provided there we replace conditions (3) and (4) by (24) and (25), respectively Further cases can be developed after the manner of the additional cases associated with 5 6 1(1) For a detailed discussion of (23) and an application, see Luke (1968a)

5 6 3 Laplace Transform and Inverse Laplace Transform of a G-Function

In 5 6 2(17), put $\sigma = 0, \tau = 1$, and $d_1 = -\alpha$ Then the Laplace transform of a G function is given by

$$\int_0^\infty e^{-\omega y} y^{-\alpha} G_{p\,q}^{m\,n}\left(zy \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dy = \omega^{\alpha-1} G_{p+1\,q}^{m\,n+1}\left(\frac{z}{\omega} \,\middle|\, \begin{matrix} \alpha, a_p \\ b_q \end{matrix}\right), \tag{1}$$

valid under the six cases stated below which may be readily found from the cases associated with 5 6 2(1) The following conditions will be needed

$$\delta = m + n - \tfrac{1}{2}(p + q) \tag{2}$$

$$R(b_j - \alpha) > -1, \qquad j = 1, 2, \ldots, m \tag{3}$$

$$a_j - b_h \text{ is not a positive integer, } \quad j = 1, 2, \ldots, n, \quad h = 1, 2, \ldots, m \tag{4}$$

$$z \neq 0 \qquad \omega \neq 0 \tag{5}$$

$$R(\alpha - a_j) > -1, \qquad j = 1, 2, \ldots, n \tag{6}$$

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right\} + \tfrac{1}{2}(q - p - 1) + (q - p)\,R(\alpha) > 0, \qquad p < q \tag{7}$$

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right\} + \tfrac{1}{2}(p + 1 - q) + (q - p)\,R(\alpha) > 0, \qquad p > q \tag{8}$$

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{p+1} b_h + \alpha\right\} > 1 \tag{9}$$

CASE 1. Let m, n, p, q, δ, ξ, and $\arg z = \arg \eta$ be as in any of the seven cases associated with 5.6.1(1). Then (1) is valid under the conditions (3)–(5) and $|\arg \omega| < \pi/2$. Equation (1) is also valid for all other possibilities of m, n, etc., not enumerated above, provided $q - p > 1$. See Case 4 below if $q = p + 1$.

CASE 2. Let m, n, p, q, δ, and $\arg z = \arg \eta$ be as in Case 1 of 5.6.2(1). Then (1) is valid under the conditions (3)–(6) and $|\arg \omega| = \pi/2$.

CASE 3. $0 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, (3)–(7), $\delta > 0$, $|\arg z| = \delta\pi$, $|\arg \omega| = \pi/2$, unless $q = p + 1$, $|\eta/\omega| = 1$, and $\arg \omega + \arg z = 0$ happen together, in which situation (1) is valid if (7) is replaced by the condition (9) and $|\arg[1 - (-)^{\delta+\frac{1}{2}}(z/\omega)]| < \pi$.

CASE 4. Let m, n, etc., be as in 5.6.2(13 or 15) with $q = p + 1$ and $\eta = z$. Then (1) is valid under the conditions (3)–(5) and $|\arg \omega| < \frac{1}{2}\pi$ when $\cos \alpha + |z/\omega| \cos \beta > 0$, $\alpha = \arg \omega$, or when $\cos \alpha + |z/\omega| \times \cos \beta = 0$ and (9) holds, provided further that in all instances $|\arg[1 - (-)^{\delta+\frac{1}{2}}(z/\omega)]| < \pi$.

The following cases, 5–7, give conditions for the validity of (1) when $p \geqslant q$.

CASE 5. Let m, n, etc., be as in Case 6 associated with 5.6.2(1), with $\eta = z$. Then (1) is valid under the conditions (3)–(5) {(3)–(6)} and $|\arg \omega| < \pi/2$ $\{|\arg \omega| = \pi/2\}$, respectively.

CASE 6. Let m, n, etc., be as in Case 8 associated with 5.6.2(1), with $\eta = z$. Then (1) is valid under the conditions (3)–(6), (8) and $|\arg \omega| = \pi/2$.

Note that when $m = 1$, (1) can be reduced to 3.6(13) and 3.6(16). The inverse Laplace transform is given by

$$z^{-\alpha} G^{m,n+1}_{p\,q+1}\left(zy \,\middle|\, \begin{matrix} a_p \\ b_q, \alpha \end{matrix}\right) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} e^{\omega z} \omega^{\alpha-1} G^{m,n}_{p,q}\left(\frac{y}{\omega} \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) d\omega, \qquad (10)$$

valid under the three cases described below. In each case $c > 0$, z is real, $z \neq 0$, $y \neq 0$. If $z \to 0$ and $R(b_h - \alpha) > 0$ for $h = 1, 2, \ldots, m$, then for $0 \leqslant n \leqslant p \leqslant q$ and $0 \leqslant m \leqslant q$, the value of the integral in (10) is nil when it converges. This follows from 5.2(7). See also (15) below. If $R(b_h) > 0$, $h = 1, 2, \ldots, m$, $z \neq 0$ and $y \to 0$, then both sides of (10) vanish. If $m = 1$, (10) can be reduced to 3.6(19).

Define δ by (2) We have need for the following conditions

$$R(\alpha - b_h) < 1, \qquad h = 1, 2, \ldots, m \tag{11}$$

$$a_j - b_h \text{ is not a positive integer}, \quad j = 1, 2, \ldots, n, \qquad h = 1, 2, \ldots, m \tag{12}$$

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right\} + \tfrac{1}{2}(p - q - 1) > (p - q)\, R(\alpha) \tag{13}$$

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h - \alpha\right\} > 1 \tag{14}$$

CASE 1 $0 \leqslant n \leqslant p \leqslant q$, $1 \leqslant m \leqslant q$, (11), (12), $\delta > 0$, $|\arg y| < \delta\pi$, $c > \max\{0, R[(-)^{m+n-p}y]\}$ if $p = q$

CASE 2 Let m, n, etc , be as in Case 6 of 5 6 2(1) Then (10) is valid under the conditions (11), (12), $\delta \geqslant 1$, $|\arg y| < (\delta - \frac{1}{2})\pi$ Under these same conditions, (10) is also valid for $m = 0$, $1 \leqslant q + 1 \leqslant n \leqslant p$

CASE 3 $1 \leqslant m \leqslant q < p$, $1 \leqslant n \leqslant p$, (11)–(13), $\delta \geqslant 1$, $|\arg y| = (\delta - \frac{1}{2})\pi$ But if $p = q + 1$ and $|zy| = 1$ happen together, then replace condition (13) by (14) and impose the restriction $|\arg(1 - (-)^{\delta+1}zy)| < \pi$ Under these same conditions, (10) is also valid for $m = 0$, $1 \leqslant q + 1 \leqslant n \leqslant p$ Further, (10) is valid for $m = 0$, $1 \leqslant q + 1 \leqslant n = p$, (12), (13), $\delta = \frac{1}{2}$, $\arg y = 0$ unless $|zy| = 1$ in which event replace (13) by (14) and require $|\arg(1 + zy)| < \pi$

We also have

$$\int_{c-i\infty}^{c+i\infty} t^{\sigma-1}\, {}_rF_s\left(\begin{matrix}1 + \sigma - c_r \\ 1 + \sigma - d_s\end{matrix}\middle|\, \eta t\right) G^{m\,n}_{p\,q}\left(\frac{z}{t}\middle|\begin{matrix}a_p \\ b_q\end{matrix}\right) dt$$

$$= \frac{2\pi i\Gamma(1 + \sigma - d_s)}{\Gamma(1 + \sigma - c_r)}\, \eta^{-\sigma}\, G^{m\,n+r}_{p+r\,q+s+1}\left(\eta z\middle|\begin{matrix}c_r, a_p \\ b_q, d_s, \sigma\end{matrix}\right),$$

$$c > 0 \quad q \geqslant 1, \quad 0 \leqslant n \leqslant p \leqslant q \quad 0 \leqslant m \leqslant q,$$

$$\eta \neq 0, \quad z \neq 0, \quad c > |z| \quad \text{if} \quad p = q,$$

$$R(c_j - b_h) < 1, \qquad j = 1, 2, \ldots, r, \qquad h = 1, 2, \ldots, m,$$

$$r = s, \qquad \eta > 0, \qquad R\left(\sigma + \sum_{j=1}^{r} d_j - \sum_{j=1}^{r} c_j - b_h\right) < 1, \qquad h = 1, 2, \ldots, m,$$

$$r = s + 1, \qquad |\arg \eta| < \pi/2, \qquad c < R(1/\eta) \tag{15}$$

This is readily proved from 5.2(1) and 3.6(26). Similarly,

$$\int_{c-i\infty}^{c+i\infty} t^{-\sigma}\, {}_sF_r\left(\begin{matrix}1+d_s-\sigma\\ 1+c_r-\sigma\end{matrix}\middle|\,\eta t\right) G^{m,n}_{p,q}\left(zt\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) dt$$

$$= \frac{2\pi i\Gamma(1+c_r-\sigma)}{\Gamma(1+d_s-\sigma)}\eta^{\sigma-1} G^{m+s,n}_{p+r+1,q+s}\left(\frac{z}{\eta}\,\middle|\,\begin{matrix}a_p, \sigma, c_r\\ d_s, b_q\end{matrix}\right),$$

$$c>0;\quad 0\leqslant n\leqslant p\leqslant q,\quad 1\leqslant m\leqslant q,$$

$$\delta^* = m+n-\tfrac{1}{2}(p+q+1)>0;\quad \eta\neq 0,\quad z\neq 0,$$

$$|\arg z|<\delta^*\pi,\quad \text{if}\quad p=q\quad \text{exclude}\quad |\arg z|$$

$$=(\delta^*-2k)\pi,\quad k=0,1,\ldots,[\delta^*/2];$$

$$R(a_j-d_h)<1,\qquad j=1,2,\ldots,n,\qquad h=1,2,\ldots,s;$$

$$r=s,\quad \eta>0,\quad R\left\{\sum_{j=1}^{r} d_j-\sum_{j=1}^{r} c_j-\sigma+a_h\right\}<1,\quad h=1,2,\ldots,n;$$

$$r=s-1,\qquad |\arg\eta|<\pi/2,\qquad c<R(1/\eta). \tag{16}$$

Another Laplace transform is

$$\int_0^\infty e^{-\beta x} G^{m,n}_{p,q}\left(\alpha x^2\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) dx = \pi^{-1/2}\beta^{-1} G^{m,n+2}_{p+2,q}\left(\frac{4\alpha}{\beta^2}\,\middle|\,\begin{matrix}0,\tfrac{1}{2},a_p\\ b_q\end{matrix}\right),$$

$$\delta>0,\quad |\arg\alpha|<\delta\pi,\quad R(\beta)>0,\quad R(b_j)>-\tfrac{1}{2},\quad j=1,2,\ldots,m, \tag{17}$$

δ as in (2), provided further that condition (4) holds. We also have the Fourier transforms

$$\int_0^\infty \cos\gamma x\, G^{m,n}_{p,q}\left(\alpha x^2\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) dx = \pi^{1/2}\gamma^{-1} G^{m,n+1}_{p+2,q}\left(\frac{4\alpha}{\gamma^2}\,\middle|\,\begin{matrix}\tfrac{1}{2},a_p,0\\ b_q\end{matrix}\right),\quad \gamma>0.$$

$$\delta>0,\quad |\arg\alpha|<\delta\pi,\quad R(b_j)>-\tfrac{1}{2},\quad j=1,2,\ldots,m,\quad R(a_j)<1,$$

$$j=1,2,\ldots,n, \tag{18}$$

$$\int_0^\infty \sin\gamma x\, G^{m,n}_{p,q}\left(\alpha x^2\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right) dx = \pi^{1/2}\gamma^{-1} G^{m,n+1}_{p+2,q}\left(\frac{4\alpha}{\gamma^2}\,\middle|\,\begin{matrix}0,a_p,\tfrac{1}{2}\\ b_q\end{matrix}\right),\quad \gamma>0,$$

$$\delta>0,\quad |\arg\alpha|<\delta\pi,\quad R(b_j)>-1,\quad j=1,2,\ldots,m,\quad R(a_j)<1,$$

$$j=1,2,\ldots,n, \tag{19}$$

again with δ as in (2) and with the additional condition (4). Other conditions for the validity of (17)–(19) can be enumerated, but we omit such details.

5 6 4 Euler and Related Transforms

$$\int_1^\infty y^{-\beta}(y-1)^{\alpha+\beta-b_1-b_2-1}\,{}_2F_1\left(\begin{matrix}\alpha-b_1,\alpha-b_2\\ \alpha+\beta-b_1-b_2\end{matrix}\middle|\,1-y\right)$$
$$\times G^{m\,n}_{p\,q}\left(zy\middle|\begin{matrix}a_p\\ \alpha,\beta\ b_3,\ \ ,b_q\end{matrix}\right)dy=\Gamma(\alpha+\beta-b_1-b_2)\,G^{m\,n}_{p\,q}\left(z\middle|\begin{matrix}a_p\\ b_q\end{matrix}\right)\qquad(1)$$

is valid under the cases described below We always take $z\neq 0$ and $m\geqslant 2$ As before define $\delta=m+n-\frac{1}{2}(p+q)$ The following conditions are needed

$$R(b_1+b_2)<R(\alpha+\beta),\quad R(a_j)<1+R(b_k),\quad k=1,2,\quad j=1,2,\quad ,n\qquad(2)$$

$$a_j-b_h\ \text{ is not a positive integer},\quad j=1,2,\quad ,n,\quad h=1,2,\quad ,m\qquad(3)$$

$$R\left\{\sum_{j=1}^{p}a_j-\sum_{j=3}^{q}b_j+(q-p)\,b_k\right\}+\tfrac{1}{2}(q-p+1)>R(\alpha+\beta),\qquad k=1,2\qquad(4)$$

CASE 1 Let m, n, etc , be as in Case 1 associated with 5 6 1(1), save that $m\geqslant 2$ Then (1) is valid under the conditions (2), (3), $\delta>0$, $|\arg z|<\delta\pi$

CASE 2 $n=0, m\geqslant 2, p+1\leqslant m\leqslant q$, (2), $\delta>0$, $|\arg z|<\delta\pi$

CASE 3 $0\leqslant n\leqslant p<q$, $2\leqslant m\leqslant q$, (2)–(4), $\delta>0$, $|\arg z|=\delta\pi$ or $0\leqslant n\leqslant p\leqslant q-2$, $m\geqslant 2$, (2)–(4), $\delta=0$, $\arg z=0$

CASE 4 $1\leqslant n\leqslant p$, $2\leqslant m\leqslant q\leqslant p-1$, (2), (3), $\delta>0$, $|\arg z|<\delta\pi$

If $q=p+2$, $q=p+1$, or $q=p$, the restrictions on $\arg z$ can be altered as in Cases 4–7 of 5 6 1(1) provided that in (2), $j=1,2,\quad ,p$

If $\beta=b_2$, (1) becomes

$$\int_1^\infty y^{-\alpha}(y-1)^{\alpha-b_1-1}\,G^{m\,n}_{p\,q}\left(zy\middle|\begin{matrix}a_p\\ \alpha,b_2,\ \ b_q\end{matrix}\right)dy=\Gamma(\alpha-b_1)\,G^{m\,n}_{p\,q}\left(z\middle|\begin{matrix}a_p\\ b_q\end{matrix}\right),\qquad(5)$$

valid under the cases above with $m\geqslant 1$

To prove (1), replace the G-function in the integrand by 5 2(1) and interchange the order of integration The inner integral can be transformed into 3 6(10) with $z=1$ and so summed by 3 13 1(1) The remaining integral is readily identified from 5 2(1) See Meijer (1940, 1941b) for examples of (1)

In a similar fashion,

$$\int_1^\infty y^{-\alpha}(y-1)^{\alpha-\beta-1}\,G^{m\,n}_{p\,q}\left(zy\middle|\begin{matrix}a_p\\ b_q\end{matrix}\right)dy=\Gamma(\alpha-\beta)\,G^{m+1\,n}_{p+1\,q+1}\left(z\middle|\begin{matrix}a_p,\alpha\\ \beta,b_q\end{matrix}\right)\qquad(6)$$

is valid under the three cases described below. Again $z \neq 0$ and $\delta = m + n - \frac{1}{2}(p + q)$. Here we need the conditions

$$R(a_j) - 1 < R(\beta) < R(\alpha), \qquad j = 1, 2, \ldots, n, \tag{7}$$

$$a_j - b_h \text{ is not a positive integer}, \quad j = 1, 2, \ldots, n, \quad h = 1, 2, \ldots, m, \tag{8}$$

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h + (q - p)\beta\right\} > \tfrac{1}{2}(p - 1 - q). \tag{9}$$

CASE 1. Let m, n, etc., be as in Case 1 associated with 5.6.1(1). Then (6) is valid under the conditions (7), (8), $\delta > 0$, $|\arg z| < \delta\pi$.

CASE 2. $n = 0$, $p + 1 \leqslant m \leqslant q$, (8), $\delta > 0$, $|\arg z| < \delta\pi$.

CASE 3. $0 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, (7)–(9), $\delta > 0$, $|\arg z| = \delta\pi$ or $0 \leqslant n \leqslant p \leqslant q - 2$, (7)–(9), $\delta = 0$, $\arg z = 0$.

Further cases may be developed as in Cases 4–7 of 5.6.1(1).

Note that if $p \geqslant q$, the left-hand side of (6) can be put in the form

$$\int_0^1 y^{\beta-1}(1 - y)^{\alpha-\beta-1} G_{q,p}^{n,m}\left(\frac{y}{z}\,\middle|\,\begin{matrix} 1 - b_q \\ 1 - a_p \end{matrix}\right) dy.$$

It is sufficient to record the result

$$\int_0^1 y^{-\alpha}(1 - y)^{\alpha-\beta-1} G_{p\,q}^{m,n}\left(zy\,\middle|\,\begin{matrix} a_p \\ b_q \end{matrix}\right) dy = \Gamma(\alpha - \beta)\, G_{p+1,q+1}^{m,n+1}\left(z\,\middle|\,\begin{matrix} \alpha, a_p \\ b_q, \beta \end{matrix}\right),$$
$$0 \leqslant n \leqslant p < q, \qquad 1 \leqslant m \leqslant q, \qquad R(\beta) < R(\alpha) < R(b_j) + 1$$
$$\text{for } j = 1, 2, \ldots, m, \tag{10}$$

provided further that (8) holds.

A generalization of (10) and so also of (6) is given by (13).

We also have the loop integrals

$$\int_{\infty}^{(1+)} y^{-\alpha}(1 - y)^{\alpha-\beta-1} G_{p,q}^{m,n}\left(zy\,\middle|\,\begin{matrix} a_p \\ b_q \end{matrix}\right) dy = -\frac{2\pi i}{\Gamma(1 + \beta - \alpha)}\, G_{p+1,q+1}^{m+1,n}\left(z\,\middle|\,\begin{matrix} a_p, \alpha \\ \beta, b_q \end{matrix}\right), \tag{11}$$

$$\int_0^{(1+)} y^{-\alpha}(1 - y)^{\alpha-\beta-1} G_{p,q}^{m,n}\left(zy\,\middle|\,\begin{matrix} a_p \\ b_q \end{matrix}\right) dy = \frac{2\pi i}{\Gamma(1 + \beta - \alpha)}\, G_{p+1,q+1}^{m,n+1}\left(z\,\middle|\,\begin{matrix} \alpha, a_p \\ b_q, \beta \end{matrix}\right), \tag{12}$$

valid under the same conditions as for (6) and (10), respectively, with the inequality $R(\alpha - \beta) > 0$ omitted.

From the work of Sharma (1964), we have a generalization of (6) and (10) Thus,

$$\int_1^\infty x^{u+\gamma-\beta-\rho-1}(x-1)^{\beta-\gamma-u}\,{}_2F_1\left(-u,\beta,\gamma,\frac{1}{x}\right)G_{p\,q}^{m\,n}\left(zx^s\middle|\begin{matrix}a_p\\b_q\end{matrix}\right)dx$$

$$=\frac{\Gamma(\gamma)\,\Gamma(\beta+1-\gamma)\,s^{u+\gamma-\beta-1}}{\Gamma(\gamma+u)}\,G_{p+2s\,q+2s}^{m+s\,n+s}\left(z\middle|\begin{matrix}c_s,a_p,d_s\\e_s,b_q,f_s\end{matrix}\right),$$

$$c_k=\frac{\rho-\gamma-u+k}{s},\qquad d_k=\frac{\rho+\beta-\gamma+k}{s},$$

$$e_k=\frac{\rho+k-1}{s},\qquad f_k=\frac{\rho-\gamma+k}{s},\qquad k=1,2,\ldots,s,\tag{13}$$

is valid under the three cases given below Let z and δ be as in the conditions for (6) We suppose that s is a positive integer and that u is a positive integer or zero Let

$$R(\beta-\gamma)>u-1,\qquad R(sa_j-\rho)<s,\qquad j=1,2,\ldots,n\tag{14}$$

Let none of the following quantities be a positive integer

$$a_j-b_h\qquad a_j-\frac{\rho+\mu-1}{s},\qquad \frac{\rho-\gamma-u+\nu}{s}-b_h,\qquad \frac{\nu-\mu-u+1-\gamma}{s},$$

$$j=1,2,\ldots,n,\qquad h=1,2,\ldots,m,\qquad \mu,\nu=1,2,\ldots,s\tag{15}$$

We also need the condition

$$R\left\{\sum_{h=1}^{p}a_h-\sum_{h=1}^{q}b_h+\frac{\rho(q-p)}{s}\right\}>\tfrac{1}{2}(p-1-q)\tag{16}$$

It is not necessary to actually enumerate the three cases for (13) as they are the same for (6) provided there the equations (7), (8), and (9) are replaced by (14), (15), and (16), respectively

If $p\geqslant q$, the left-hand side of (13) may be expressed as

$$\int_0^1 y^{\rho-1}(1-y)^{\beta-\gamma-u}\,{}_2F_1(-u,\beta,\gamma,y)\,G_{q\,p}^{n\,m}\left(\frac{y^s}{z}\middle|\begin{matrix}1-b_q\\1-a_p\end{matrix}\right)dy$$

and so it is sufficient to record the formula

$$\int_0^1 x^{\rho-1}(1-x)^{\beta-\gamma-u}\,{}_2F_1(-u,\beta,\gamma,x)\,G_{p\,q}^{m\,n}\left(zx^s\middle|\begin{matrix}a_p\\b_q\end{matrix}\right)dx$$

$$=\frac{\Gamma(\gamma)\,\Gamma(\beta+1-\gamma)\,s^{u+\gamma-\beta-1}}{\Gamma(\gamma+u)}\,G_{p+2s\,q+2s}^{m+s\,n+s}\left(z\middle|\begin{matrix}g_s,a_p,h_s\\i_s,b_q,j_s\end{matrix}\right)$$

$$g_k = \frac{k-\rho}{s}, \qquad h_k = \frac{\gamma-\rho+k-1}{s},$$

$$i_k = \frac{\gamma+u-\rho+k-1}{s}, \qquad j_k = \frac{\gamma-\rho-\beta+k-1}{s}, \qquad k = 1, 2, \ldots, s,$$

$$0 \leqslant n \leqslant p < q, \quad 1 \leqslant m \leqslant q, \quad R(\beta-\gamma) > u-1, \quad R(\rho+sb_h) > 0,$$
$$h = 1, 2, \ldots, m, \tag{17}$$

provided further that none of the following quantities are positive integers:

$$a_j - b_h, \quad a_j - \frac{\gamma-\rho+u+\mu-1}{s}, \quad \frac{v-\rho}{s} - b_h, \quad \frac{v-\mu-u+1-\gamma}{s}$$

where h, j, μ, and v take on the values as in (15).

An outline of the proof for (17) follows. Replace the G-function in the integrand by 5.2(1) and interchange the order of integration. The inner integral is readily evaluated from 3.6(10) and 3.13.3(2). The result is a product of gamma functions and a typical factor is $\Gamma(\rho+st)$ where t is the variable of integration for the outer integral. Now use the multiplication formula for gamma functions, 2.3(1), and the outer integral is easily identified from 5.2(1).

Two special cases of (17) which were proved by Saxena (1962) follow. With $x = \sin^2\theta$, $\beta = u-1$, $\gamma = \frac{3}{2}$, and the aid of 6.2.1(12),

$$\int_0^\pi \sin^{2\gamma-2}\theta \sin(2u-1)\theta\, G^{m,n}_{p,q}\left(z\sin^{2s}\theta \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) d\theta$$
$$= \left(\frac{\pi}{s}\right)^{1/2} G^{m-s,n-s}_{p-2s,q-2s}\left(z \,\middle|\, \begin{matrix} g_s, a_p, h_s \\ i_s, b_q, j_s \end{matrix}\right). \tag{18}$$

and with x as above, $\beta = u$, $\gamma = \frac{1}{2}$, and the aid of 6.2.1(11),

$$\int_0^\pi (\sin^{2\gamma-1}\theta, 2)\cos u\theta\, G^{m,n}_{p,q}\left(z\sin^{2s}\frac{\theta}{2} \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) d\theta$$
$$= \left(\frac{\pi}{s}\right)^{1/2} G^{m-s,n-s}_{p-2s,q-2s}\left(z \,\middle|\, \begin{matrix} g_s, a_p, h_s \\ i_s, b_q, j_s \end{matrix}\right). \tag{19}$$

Sharma (1964) has also proved

$$\int_1^\infty x^{-\beta}(x-1)^{\beta-\alpha-1}\, {}_2F_1(\alpha, \beta; \gamma; 1/x)\, G^{m,n}_{p,q}\left(z(x-1)^s \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) dx$$
$$= \frac{(2\pi)^{1-s}\,\Gamma(\gamma)\, s^{\beta-\alpha-1}}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}\, G^{m-s,n-2s}_{p-2s,q-2s}\left(z \,\middle|\, \begin{matrix} r_s, t_s, a_p \\ e_s, b_q, f_s \end{matrix}\right),$$
$$r_k = e_k - (1-\beta)s, \qquad t_k = f_k + \alpha s, \qquad k = 1, 2, \ldots, s, \tag{20}$$

where e_k and f_k are defined in (13) To discuss conditions of validity, let

$$\begin{gathered} R(\beta) > R(\rho) > 0, \qquad R(\gamma - \alpha - \rho) > 0, \qquad R(\gamma - \alpha - \beta) > 0, \\ R(sb_h + \beta - \rho) > 0, \qquad h = 1, 2, \ldots, m, \\ R(sa_j - \rho) < s, \qquad j = 1, 2, \ldots, n, \end{gathered} \tag{21}$$

and suppose that none of the following quantities are positive integers:

$$\begin{gathered} a_j - b_h, \qquad a_j - \frac{\rho + \mu - 1}{s}, \qquad \frac{\rho - \gamma + \alpha + \nu}{s} - b_h, \\ \frac{\rho - \beta + \sigma}{s} - b_h, \qquad \frac{\alpha - \gamma + 1 + \nu - \sigma}{s}, \qquad \frac{\sigma - \mu + 1 - \beta}{s}, \\ j = 1, 2, \ldots, n, \qquad h = 1, 2, \ldots, m, \qquad \mu, \nu, \sigma = 1, 2, \ldots, s \end{gathered} \tag{22}$$

Then conditions of validity for (20) are the same as for (6) provided that there the equations (7), (8), and (9) are replaced by (21), (22), and (16), respectively. In all these conditions $p \leqslant q$ If $p \geqslant q$, the left-hand side of (20) may be written as

$$\int_0^1 y^{\rho-1}(1-y)^{\beta-\rho-1}\,{}_2F_1(\alpha, \beta, \gamma, y)\, G_{q\,p}^{m\,n}\left(\frac{y^s(1-y)^{-s}}{z}\,\middle|\,\begin{matrix}1-b_q\\1-a_p\end{matrix}\right) dy.$$

It is therefore sufficient to record the result

$$\begin{aligned} &\int_0^1 x^{\rho-1}(1-x)^{\beta-\rho-1}\,{}_2F_1(\alpha, \beta, \gamma, x)\, G_{p\,q}^{m,n}\left(zx^s(1-x)^{-s}\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right) dx \\ &\quad = \frac{(2\pi)^{1-s}\,\Gamma(\gamma)\,s^{\beta-\alpha-1}}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}\, G_{p+2s\,q+2s}^{m+2s,n+s}\left(z\,\middle|\,\begin{matrix}g_s, a_p, h_s\\w_s, y_s, b_q\end{matrix}\right), \\ &w_k = \frac{\beta - \rho + k - 1}{s}, \quad y_k = \frac{\gamma - \alpha - \rho + k - 1}{s}, \quad k = 1, 2, \ldots, s, \end{aligned} \tag{23}$$

where g_k and h_k are defined in (17), provided

$$\begin{gathered} 1 \leqslant n \leqslant p < q, \quad 1 \leqslant m \leqslant q, \quad \delta = m + n - \tfrac{1}{2}(p+q) > 0, \quad |\arg z| < \delta\pi, \\ R(\beta) > R(\rho) > 0, \quad R(\gamma - \alpha - \rho) > 0, \quad R(\gamma - \alpha - \beta) > 0, \\ R(\rho + sb_h) > 0, \qquad h = 1, 2, \ldots, m, \\ R(\rho - \beta + sa_j) < s, \qquad j = 1, 2, \ldots, n, \end{gathered} \tag{24}$$

and that none of the following quantities are positive integers

$$a_j - b_h, \qquad a_j - \frac{\beta - \rho - 1 + \nu}{s}, \qquad a_j - \frac{\gamma - \alpha - \rho - 1 + \sigma}{s}, \qquad \frac{\mu - \rho}{s} - b_h,$$
$$\frac{\mu - \nu + 1 - \beta}{s}, \qquad \frac{\mu - \sigma + \alpha - \gamma + 1}{s} \quad \text{with } j, h, \mu, \nu, \text{ and } \sigma \text{ as in (22)} \tag{25}$$

Proof of (23) is much like that of (17) and we omit details.

Two special cases of (23) are of interest. If $\beta = \alpha - \frac{1}{2}, \gamma = 2\alpha$, and we use 6.2.1(5), then

$$\int_0^1 x^{\rho-1}(1-x)^{\alpha-\rho-3/2}[1+(1-x)^{1/2}]^{1-2\alpha}\, G_{p,q}^{m,n}\left(zx^s(1-x)^{-s}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)dx$$

$$= \frac{(2\pi)^{1-s}(\alpha-\frac{1}{2})\,s^{-3/2}}{\pi^{1/2}}\, G_{p+2s,q+2s}^{m+2s,n+s}\left(z\,\middle|\,\begin{matrix}g_s\,,\,a_p\,,\,h_s\\ w_s\,,\,y_s\,,\,b_q\end{matrix}\right). \tag{26}$$

Again in (23), replace α by $\alpha - \frac{1}{2}$, then set $\beta = \alpha$ and $\gamma = 2\alpha$, and use 6.2.1(5). Then

$$\int_0^1 x^{\rho-1}(1-x)^{\alpha-\rho-1}[1+(1-x)^{1/2}]^{1-2\alpha}\, G_{p,q}^{m,n}\left(zx^s(1-x)^{-s}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)dx$$

$$= \frac{(2\pi)^{1-s}}{(\pi s)^{1/2}}\, G_{p+2s,q+2s}^{m+2s,n+s}\left(z\,\middle|\,\begin{matrix}g_s\,,\,a_p\,,\,h_s\\ w_s'\,,\,y_s'\,,\,b_q\end{matrix}\right),$$

$$w_k' = \frac{\alpha-\rho+k-1}{s}, \qquad y_k' = \frac{\alpha-\rho+k-\frac{1}{2}}{s}, \qquad k = 1, 2, \ldots, s. \tag{27}$$

The substitution $x = \operatorname{sech}^2\theta$ transforms the left-hand side of the latter into

$$2\int_0^\infty e^{-\theta(2\alpha-1)}(\sinh\theta)^{2\alpha-2\rho-1}\, G_{p,q}^{m,n}\left(z\operatorname{csch}^{2s}\theta\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)d\theta. \tag{28}$$

From the work of Saxena (1963), we have

$$\int_0^\infty t^{\lambda-1}(1+t)^{-1/2}[t^{1/2}+(1+t)^{1/2}]^{2\mu}\, G_{p,q}^{m,n}\left(zt^r\left\{t^{1/2}+(1+t)^{1/2}\right\}^{2s}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)dt$$

$$= \frac{(2\pi)^{1/2-r-s}\,2^{1/2}\,r^{2\lambda-1/2}}{(r+s)^{\lambda+\mu}(r-s)^{\lambda-\mu}}$$

$$\times\, G_{p+2r,q+2r}^{m+r+s,n+2r}\left(\frac{zr^{2r}}{(r+s)^{r+s}(r-s)^{r-s}}\,\middle|\,\begin{matrix}c_{2r}\,,\,a_p\\ d_{r+s}\,,\,b_q\,,\,e_{r-s}\end{matrix}\right), \qquad r > s, \tag{29}$$

$$= \frac{(2\pi)^{1/2-2r}\,2^{1/2}\,r^{2\lambda-1/2}}{(r+s)^{\lambda+\mu}(s-r)^{\lambda-\mu}}$$

$$\times\, G_{p+r+s,q+r+s}^{m+r+s,n+2r}\left(\frac{zr^{2r}(s-r)^{s-r}}{(s+r)^{s+r}}\,\middle|\,\begin{matrix}c_{2r}\,,\,a_p\,,\,f_{s-r}\\ d_{r+s}\,,\,b_q\end{matrix}\right), \qquad r < s, \tag{30}$$

$$= \frac{\pi^{1-2r}\,r^{\lambda-\mu-1/2}}{2^{\lambda+\mu+2r-3/2}\,\Gamma(\frac{1}{2}+\lambda-\mu)}\, G_{p+2r,q+2r}^{m+2r,n+2r}\left(\frac{z}{2^{2r}}\,\middle|\,\begin{matrix}c_{2r}\,,\,a_p\\ d_{2r}\,,\,b_q\end{matrix}\right), \qquad r = s, \tag{31}$$

where r and s are positive integers [s can be zero in (29)] and

$$\begin{aligned} c_\nu &= \frac{\nu - 2\lambda}{2r}, & \nu &= 1, 2, \ldots, 2r, \\ d_\sigma &= \frac{2\sigma - 1 - 2\mu - 2\lambda}{2r + 2s}, & \sigma &= 1, 2, \ldots, r + s, \\ e_\omega &= \frac{2\omega - 1 + 2\mu - 2\lambda}{2r - 2s}, & \omega &= 1, 2, \ldots, r - s, \\ f_\epsilon &= \frac{2\epsilon - 1 + 2\lambda - 2\mu}{2s - 2r}, & \epsilon &= 1, 2, \ldots, s - r \end{aligned} \tag{32}$$

To simplify discussion of cases for the validity of (29)–(31), we suppose that $z \neq 0$, $\delta = m + n - \frac{1}{2}(p + q)$ We have need for the following conditions

$$\begin{aligned} R(rb_h + \lambda) &> 0, & h &= 1, 2, \ldots, m, \\ R(\lambda + \mu + (r + s)\, a_j) &< r + s + \tfrac{1}{2}, & j &= 1, 2, \ldots, n \end{aligned} \tag{33}$$

None of the quantities

$$\begin{aligned} &a_j - b_h, \quad a_j - d_\sigma, \quad j = 1, 2, \ldots, n, \quad h = 1, 2, \ldots, m, \\ &c_\nu - b_h, \quad c_\nu - d_\sigma, \quad \sigma = 1, 2, \ldots, r + s, \quad \nu = 1, 2, \ldots, 2r, \end{aligned} \tag{34}$$

is a positive integer

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h + \frac{(q - p)}{r + s}\left(\tfrac{1}{2} - \lambda - \mu\right)\right\} > \tfrac{1}{2}(p - 1 - q) \tag{35}$$

$$R\left\{\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h + \frac{(p - q)\lambda}{r}\right\} > \tfrac{1}{2}(p - 1 - q) \tag{36}$$

Three cases for the validity of (29)–(31) follow

CASE 1 Let m, n, etc, be as in Cases 1, 2, and 3 of 5 6 1(1) Then (29)–(31) are valid as for the latter cases with $\eta = z$, provided there we replace condition (3) by (33), (34) and (4) by (35), respectively

CASE 2 $1 \leqslant m \leqslant q < p$, $1 \leqslant n \leqslant p$ or $q \geqslant 1$, $0 \leqslant m \leqslant q$, $1 \leqslant n \leqslant p = q + 1$, $\delta > 0$, $|\arg z| < \delta\pi$

CASE 3 $0 \leqslant m \leqslant q < p$, $1 \leqslant n \leqslant p$, (33), (34), (36), $\delta > 0$, $|\arg z| < \delta\pi$, or $0 \leqslant m \leqslant q \leqslant p - 2$, (33), (34), (36), $\delta = 0$, $\arg z = 0$

Equations (29)–(31) are proved from 5.2(1), 2.3(1), and the formula

$$\int_0^\infty t^{\lambda-1}(1+t)^{-1/2}[t^{1/2}+(1+t)^{1/2}]^{2\mu}\,dt = \frac{\Gamma(2\lambda)\,\Gamma(\frac{1}{2}-\lambda-\mu)}{2^{2\lambda-1}\,\Gamma(\frac{1}{2}+\lambda-\mu)},$$

$$R(\lambda) > 0, \qquad R(\lambda+\mu) < \tfrac{1}{2}, \tag{37}$$

which may be proved from 6.2.1(5), 3.6(10), and 3.13.1(1).

5.6.5. Integral Transform Pairs Involving the G-Function

The functions $f(x)$ and $g(x)$ are said to form a pair of Fourier kernels if

$$g(x) = \int_0^\infty k(xy)f(y)\,dy, \qquad f(x) = \int_0^\infty h(xy)\,g(y)\,dy \tag{1}$$

are simultaneously satisfied. The kernels are called symmetric if $k(x) = h(x)$, otherwise they are unsymmetric. Narain (1962, 1963) has shown that

$$k(x) = 2\gamma x^{\nu-1/2}\,G^{m,p}_{p+q,m+n}\left(x^{2\nu}\,\middle|\,\begin{matrix} a_p\,, b_q \\ c_m\,, d_n\end{matrix}\right), \tag{2}$$

$$h(x) = 2\gamma x^{\nu-1/2}\,G^{n,q}_{p+q,m+n}\left(x^{2\nu}\,\middle|\,\begin{matrix} -b_q\,, -a_p \\ -d_n\,, -c_m\end{matrix}\right), \tag{3}$$

are unsymmetrical kernels. If $p = q$, $m = n$, $a_j + b_j = 0$ for $j = 1, 2,\ldots, p$ and $c_h + d_h = 0$ for $h = 1, 2,\ldots, m$, then the kernels are symmetric, a result previously derived by C. Fox (1961). For conditions of validity, see the sources cited.

The following pair of transforms are due to Wimp (1964). If

$$g(x) = \int_0^\infty G^{m,n+2}_{p+2,q}\left(t\,\middle|\,\begin{matrix} 1-\nu+ix,\ 1-\nu-ix,\ a_p \\ b_q\end{matrix}\right) f(t)\,dt, \tag{4}$$

then

$$f(x) = i/\pi \int_0^\infty te^{-\nu\pi i}\{e^{\pi t}A(xe^{i\pi}\mid \nu+it, \nu-it) - e^{-\pi t}A(xe^{i\pi}\mid \nu-it, \nu+it)\}\,g(t)\,dt,$$

$$A(z\mid\alpha,\beta) = G^{q-m,p-n+1}_{p+2,q}\left(z\,\middle|\,\begin{matrix} -a_{n+1}\,, -a_{n+2}\,,\ldots, -a_p\,, \alpha, -a_1\,, -a_2\,,\ldots, -a_n\,, \beta \\ -b_{m+1}\,, -b_{m+2}\,,\ldots, -b_q\,, -b_1\,, -b_2\,,\ldots, -b_m\end{matrix}\right). \tag{5}$$

This set of relations contains some known important special cases. Thus the Kontorovich–Lebedev transform pair is

$$g(x) = (2/\pi^2)\,x\sinh \pi x \int_0^\infty t^{-1}K_{ix}(t)\,f(t)\,dt, \tag{6}$$

$$f(x) = \int_0^\infty K_{it}(x)\,g(t)\,dt, \tag{7}$$

and the generalized Mehler transform pair is

$$g(x) = (x/\pi)\,\Gamma(\tfrac{1}{2} - k + ix)\,\Gamma(\tfrac{1}{2} - k - ix)\sinh \pi x \int_1^\infty P^k_{ix-1/2}(t) f(t)\,dt \tag{8}$$

$$f(x) = \int_0^\infty P^k_{i\ 1\ 2}(x) g(t)\,dt \tag{9}$$

In (6) (7) $K(t)$ is the modified Bessel function see 6 2 7(8) and in (8) (9) $P_\nu^\mu(t)$ is the Legendre function see 6 2 3 For proof and other special cases of (4) (5) see Wimp (1964) For rather extensive tables of the transform pairs (6) (7) and (8) (9) see Erdelyi *et al* (1954) and Oberhettinger and Higgins (1961)

5 7 Asymptotic Expansion of $G^{q\ 1}_{p,q}(z)$ and $G^{q\ 0}_{p\ q}(z)$ for Large z

To simplify the exposition in this section as well as in Sections 5 8–5 10 it is convenient to introduce some notation We let

$$m + n \quad p = \tau \qquad q \quad m - n - \nu \tag{1}$$

$$m + n \quad \tfrac{1}{2}(p + q) = \tfrac{1}{2}(\tau \quad \nu) \quad \rho \qquad \tfrac{3}{2}q - \tfrac{1}{2}p - m \quad n + 2 = \eta \qquad q \quad p = \sigma$$
$$\epsilon = \tfrac{1}{2} \quad \text{if} \quad \sigma = 1 \qquad \epsilon = 1 \quad \text{if} \quad \sigma > 1 \tag{2}$$

Certain restrictions on the parameters a_h and b_h of the G function enter the hypotheses of the theorems These are called (A), (B), and (C) and are as follows

$$\text{(A)} \qquad a_j - b_h \neq 1\ 2\ 3 \qquad \text{for} \quad j = 1\ 2 \quad n \quad h = 1\ 2 \quad m \tag{3}$$

$$\text{(B)} \qquad a_j - a_t \neq 0\ \pm 1\ \pm 2 \qquad \text{for} \quad j\ t = 1\ 2 \quad n \quad j \neq t \tag{4}$$

$$\text{(C)} \qquad a\ - a_h \neq 0\ \pm 1\ \pm 2 \qquad \text{for} \quad j\ h - 1\ 2 \quad p \quad j \neq h \tag{5}$$

Actually (B) can always be deleted from our assumptions provided we understand that when (B) does not hold a passage to the limit is required as in the discussion surrounding 5 1(27) A similar remark also holds for (C)

Define

$$G^{q\ 1}_{p\ q}(z \| a_t) = G^{q\ 1}_{p\ q}\left(z \middle| \begin{matrix} a_1 & a_1 & & a_{t-1} & a_{t+1} & & a_p \\ & & & b_q & & & \end{matrix}\right) \tag{6}$$

Further with $1 \leqslant t \leqslant p < q$ let us formally write

$$E_{p\ q}(z \| a_t) = \frac{z^{a\ \ 1}\Gamma(1 + b_q - a_t)}{\Gamma(1 + a_p - a_t)}\,{}_qF_{p-1}\left(\begin{matrix} 1 + b_q - a_t \\ 1 + a_p - a_t^* \end{matrix} \middle| -\frac{1}{z}\right) \tag{7}$$

provided condition (A) [see (3)] holds with $j = t$ and $m = q$

Theorem 1. *If* $1 \leqslant t \leqslant p < q$, (A), *then*

$$G^{q,1}_{p,q}(z \,\|\, a_t) \sim E_{p,q}(z \,\|\, a_t),$$

$$|z| \to \infty, \qquad |\arg z| \leqslant (\tfrac{1}{2}\sigma + 1)\pi - \delta, \qquad \delta \geqslant 0. \tag{8}$$

Theorem 2. *If* $1 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, (A) *and* (B), *then with* $\rho > 0$,

$$G^{m,n}_{p,q}(z) \sim \sum_{t=1}^{n} \exp[-i\pi(\nu + 1)\, a_t]\, \Delta^{m,n}_{q}(t)\, E_{p,q}(z \exp[i\pi(\nu + 1)] \,\|\, a_t),$$

$$|z| \to \infty, \qquad |\arg z| \leqslant \rho\pi - \delta, \qquad \delta \geqslant 0, \tag{9}$$

where $\Delta^{m,n}_{q}(t)$ *is defined by* 5.9.2(1).

Formulas (8) and (9) are proved by evaluating the residues of the integrand in 5.2(1) which lie within the path described by 5.2(4). Note that the expressions on the right-hand sides of (9) and 5.2(11) are the same.

Theorem 3. *If* $m + n \geqslant p + 1$, (A) *and* (B), *and* $|\arg z| < (m + n - p)\pi$, *then* $G^{m,n}_{p,p}(z)$ *can be continued analytically outside the unit circle by the expansion*

$$G^{m,n}_{p,p}(z) = \sum_{t=1}^{n} \exp[-i\pi(p - m - n + 1)\, a_t]\, \Delta^{m,n}_{p}(t)$$

$$\times\, E_{p,p}(z \exp[i\pi(p - m - n + 1)] \,\|\, a_t). \tag{10}$$

Theorem 4. *If* (A) *holds for* $n = 1$ *and* $m = p$, *and* $|\arg z| < \pi$, *then* $G^{p,1}_{p,p}(z \,\|\, a_t)$ *can be continued analytically outside the unit circle by the expression*

$$G^{p,1}_{p,p}(z \,\|\, a_t) = E_{p,p}(z \,\|\, a_t). \tag{11}$$

Theorem 3 has been proved in 5.3. Theorem 4 is the special case $n = 1$ of Theorem 3.

Theorem 5.

$$G^{q,0}_{p,q}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) \sim H_{p,q}(z),$$

$$|z| \to \infty, \qquad |\arg z| \leqslant (\sigma + \epsilon)\pi - \delta, \qquad \delta > 0, \tag{12}$$

where σ and ϵ are defined in (1), (2) *and where for brevity we write*

$$H_{p,q}(z) = \frac{(2\pi)^{(\sigma-1)/2}}{\sigma^{1/2}} \exp\{-\sigma(z)^{1/\sigma}\} z^{\theta} \sum_{k=0}^{\infty} M_k(z)^{-k/\sigma},$$

$$M_0 = 1, \qquad \sigma\theta = \{\tfrac{1}{2}(1-\sigma) + \Xi_1 - \Lambda_1\},$$

$$\Xi_1 = \sum_{h=1}^{q} b_h, \qquad \Lambda_1 = \sum_{h=1}^{p} a_h \tag{13}$$

The M_k's are independent of z and can be found from the theory in 5 11 5 For $k = 1$, we have

$$M_1 = (\Lambda_2 - \Xi_2) - \frac{(\Lambda_1 - \Xi_1)}{2\sigma}[\sigma(\Lambda_1 + \Xi_1) + (\Lambda_1 - \Xi_1)] - \frac{(\sigma^2 - 1)}{24\sigma}, \tag{14}$$

where

$$\Xi_2 = \sum_{h=2}^{q} \sum_{k=1}^{h-1} b_h b_k, \qquad \Lambda_2 = \sum_{h=2}^{p} \sum_{k=1}^{h-1} a_h a_k \tag{15}$$

We sketch the proof of (12) as follows Let $G^{q\,0}_{p\,q}(z)$ be defined by 5 2(1, 2) Note that except for z^s, the integrand of this representation is ${}_pk_q(z)$ of 2 11(25) with an appropriate change of notation There replace z by s, q by $q-1$, $1-b_{j-1}$ by b_j for $j = 1, 2, \ldots, q$, and $1-a_j$ by a_j for $j = 1, 2, \ldots, p$ It follows that

$$G^{q\,0}_{p\,q}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = (2\pi i)^{-1}(2\pi)^{(\sigma-1)/2}\sigma^{1/2-\sigma\theta}\left[\sum_{j=0}^{N-1}(-)^j h_j^* \int_L \sigma^{\sigma s} z^s \Gamma(\sigma\theta - \sigma s - j)\, ds\right.$$

$$\left. + \int_L \sigma^{\sigma s} z^s O(\Gamma(\sigma\theta - \sigma s - N))\, ds\right],$$

where h_j^* comes from h_j when the appropriate substitutions are made Now

$$(2\pi i)^{-1}\int_L \sigma^{\sigma s} z^s \Gamma(\sigma\theta - \sigma s - j)\, ds = \frac{(\sigma z^{1/\sigma})^{\sigma\theta - j}}{\sigma} \exp\{-\sigma z^{1/\sigma}\}$$

in view of 5 2(1, 14) and 5 4(4) Hence,

$$G^{q\,0}_{p\,q}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) = \frac{(2\pi)^{(\sigma-1)/2}}{\sigma^{1/2}} \exp\{-\sigma(z)^{1/\sigma}\} z^{\theta}\left[\sum_{j=0}^{N-1} \frac{(-)^j h_j^*(z)^{j/\sigma}}{\sigma^j} + R_N\right]$$

It may be shown that $R_N = O(z^{-N/\sigma})$ for $\arg z$ as in (12) and so we arrive at (12), (13) For complete details of the proof, see Braaksma (1963)

5.8. Differential Equation for $G_{p,q}^{m,n}(z)$

From 5.2(1) or 5.2(7), it follows that $y(z) = G_{p,q}^{m,n}(z \mid {}^{a_p}_{b_q})$ satisfies

$$[(-)^\tau z(\delta - a_p + 1) - (\delta - b_q)]\, y(z) = 0, \qquad \delta = z\, d/dz, \tag{1}$$

where τ is defined by 5.7(1). This equation is of order $\max(p, q)$ and in view of 5.4(3), we can suppose that $p \leqslant q$. If $p < q$, the only singularities of (1) are at $z = 0$, a regular singularity, and at $z = \infty$, an irregular singularity. If $p = q$, $z = 0$ and $z = \infty$ are regular singularities, and in addition there is a regular singularity at $z = (-)^\tau$. The solutions of (1) in the neighborhood of the singularities $z = 0$ and $z = \infty$ have been fully investigated by Meijer (1946, pp. 344–356). No fundamental system for the neighborhood of $z = (-)^\tau$ has been given in the literature. In this connection, see Nörlund (1955).

In the neighborhood of $z = 0$, the q functions

$$y_h(z) = \exp[i\pi(\tau + 1)\, b_h]\, G_{p,q}^{1,p}\left(z \exp[-i\pi(\tau + 1)] \,\middle|\, {}^{a_p}_{b_h, b_1, \ldots, b_{h-1}, b_{h+1}, \ldots, b_q}\right),$$
$$h = 1, 2, \ldots, q, \tag{2}$$

form a fundamental system of solutions for (1) provided that no two of the b_h terms $h = 1, 2, \ldots, m$, differ by an integer or zero. This condition is really not essential, see the discussion surrounding 5.2(9). Clearly

$$y_h(z) = \frac{\Gamma(1 + b_h - a_p)}{\Gamma(1 + b_h - b_q)}\, z^{b_h}\, {}_pF_{q-1}\left({}^{1 + b_h - a_p}_{1 + b_h - b_q^*} \,\middle|\, (-)^\tau z\right),$$
$$p \leqslant q - 1, \quad \text{or} \quad p = q \quad \text{and} \quad |z| < 1. \tag{3}$$

It follows that $G_{p,q}^{m,n}(z)$ is a linear combination of the functions (2). The pertinent expression is given by 5.2(18).

Next we consider solutions of (1) in the vicinity of the irregular singularity $z = \infty$. We distinguish two cases.

CASE 1. $p < q$. For every value of $\arg z$, it is possible to find integers λ and ω such that

$$|\arg z + (\nu - 2\lambda + 1)\pi| < (\tfrac{1}{2}\sigma + 1)\pi, \tag{4}$$

$$|\arg z + (\nu - 2\psi)\pi| < (\sigma + \epsilon)\pi, \qquad \psi = \omega, \omega + 1, \ldots, \omega + \sigma - 1, \tag{5}$$

where ν, σ, and ϵ are defined in 5.7(1, 2).

If condition (C) [see 5.7(5)] holds, then the p functions

$$G_{p,q}^{q,1}(z \exp[i\pi(\nu - 2\lambda + 1)] \| a_t), \qquad t = 1, 2, \ldots, p \tag{6}$$

satisfy (1) From 5 7(7, 8) these functions tend algebraically to zero or to infinity as $|z| \to \infty$, and further, for large values of $|z|$, are linearly independent because of condition (C) If (C) does not hold, we can get independent solutions via limit processes, as previously explained In this event, the algebraic character of the solutions is not altered as $|z| \to \infty$ Thus (6) gives rise to p solutions in the region of $z = \infty$ To get the remaining $\sigma = q - p$ solutions, consider

$$G^{q\ 0}_{p\ q}\left(z \exp[i\pi(\nu - 2\psi)] \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) \qquad \psi = \omega, \omega + 1, \ \ , \omega + \sigma - 1 \tag{7}$$

These functions also satisfy (1) and, in virtue of 5 7(12) and (5), tend exponentially to zero or to infinity as $|z| \to \infty$ Furthermore, the solutions (7) are linearly independent We have thus proved that if $q > p$ and the conditions (C), (4), (5) hold, then a fundamental system of (1) valid near $z = \infty$ is formed by the p functions (6) and the $\sigma = q - p$ functions (7)

CASE 2 $p = q$ Again we assume that the a_h's satisfy condition (C) We also suppose that

$$\arg z - \tau\pi \neq \pm 2k\pi, \qquad k = 0\ 1\ 2 \tag{8}$$

Then from (4) we can find an integer λ such that

$$|\arg z - (\tau + 2\lambda - 1)\pi| < \pi \tag{9}$$

In this case, (6) with $q = p$ satisfies (1) Further, because of (C) and 5 7(7, 11), these p solutions are linearly independent

There may occur in a formula not the p functions (6), or the same set of functions with $q = p$, but only the first n of these, namely,

$$G^{q\ 1}_{p\ q}(z \exp[i\pi(\nu - 2\lambda + 1)] \ a_t), \qquad t = 1\ 2, \ \ , n, \tag{10}$$

or the latter set with $q = p$ For these to be linearly independent, condition (C) can be replaced by the less stringent condition (B)

If we have to do with less than the σ functions $G^{q\ 0}_{p\ q}(z)$, namely, the $\alpha(\alpha < \sigma)$ functions

$$G^{q\ 0}_{p\ q}(z \exp[i\pi(\nu - 2\psi)]), \qquad \psi = \omega, \omega + 1, \quad \omega + \alpha - 1, \tag{11}$$

then the condition (5) need not be satisfied for $\psi = \omega, \omega + 1, \ \ , \omega + \sigma - 1$, but only for $\psi = \omega, \omega + 1, \ \ , \omega + \alpha - 1$

5.9. Series of G-Functions

5.9.1. Introduction

The multiplication theorems of 5.5. permit for the expansion of G-functions in series of G-functions. Another group of formulas of this kind is taken up in Chapter IX. Here we delineate four important expansions which express $G_{p,q}^{m,n}(z)$ as a finite sum of G-functions with the same p, q but with $m = q$ and $n = 0$ or $n = 1$. For the most part, proofs are omitted. For complete details of the proof for the expansion theorems and generalizations of same, see Meijer (1946).

Corresponding to each expansion theorem there is a companion theorem (in one instance there is more than one companion theorem) which gives the conditions under which the expansion theorem expresses $G_{p,q}^{m,n}(z)$ in terms of fundamental solutions of 5.8(1) valid near $z = \infty$. These data coupled with the asymptotic expansions of $G_{p,q}^{q,1}(z)$ and $G_{p,q}^{q,0}(z)$ [see 5.7(8, 12)] lead to the asymptotic expansions for $G_{p,q}^{m,n}(z)$ which are developed in 5.10. The asymptotic expansion for the ${}_pF_q$ is treated in 5.11.

5.9.2. Notation

We freely use the notation of 5.7(1–6). Other notation follows. Let

$$\Delta^{m,n}_{q}(t) = (-)^{\nu+1} \frac{\prod_{j=1}^{n} \Gamma(a_t - a_j)^* \Gamma(1 + a_j - a_t)}{\prod_{j=m+1}^{q} \Gamma(a_t - b_j)\, \Gamma(1 + b_j - a_t)}. \tag{1}$$

Where no confusion can result, we simply write $\Delta(t)$. This practice of omitting super and subscripts when appropriate is usually adopted in the definitions which follow. Thus we define

$$A = A^{m,n}_{q} = (-)^{\nu} (2\pi i)^{-\nu} \exp\left\{ i\pi \left(\sum_{j=1}^{n} a_j - \sum_{j=m+1}^{q} b_j \right) \right\}. \tag{2}$$

If in the right-hand side of (2), i is replaced by $-i$, the resulting expression is notated as $\bar{A}^{m,n}_{q}$ or simply as $\bar{A}$ when appropriate. The bar over a letter convention also applies to the quantities defined in (3)–(5), (13)–(16) below. Thus,

$$B = B_{p}^{m,n} = (-)^{\tau} (2\pi i)^{\tau} \exp\left\{ i\pi \left(\sum_{j=1}^{m} b_j - \sum_{j=n+1}^{p} a_j \right) \right\}, \tag{3}$$

$$\bar{B} = \bar{B}_{p}^{m,n} = (2\pi i)^{\tau} \exp\left\{ -i\pi \left(\sum_{j=1}^{m} b_j - \sum_{j=n+1}^{p} a_j \right) \right\}. \tag{4}$$

In these and all other definitions, it is supposed that the parameters are such that the definitions make sense Let

$$\frac{\prod_{j=m+1}^{q}[1 - x\exp(2i\pi b_j)]}{\prod_{j=1}^{n}[1 - x\exp(2i\pi a_j)]} = \sum_{k=0}^{\infty} \Omega^{m\,n}_{q}(k)\, x^k,$$

$$|x\exp(2i\pi a_j)| < 1, \qquad \Omega^{m\,n}_{q}(k) = \Omega(k) \tag{5}$$

We next present some developments to facilitate the computation of $\Omega(k)$ Put

$$\xi_0 = 1 \qquad \xi_1 = \sum_{j=1}^{n} \exp(2i\pi a_j), \qquad \xi_2 = \sum_{1\leqslant j<k\leqslant n} \exp[2i\pi(a_j + a_k)], \quad ,$$
$$\xi_r = \sum_{1\leqslant h_1<h_2<\ \ <h_{r-1}\leqslant h_r}^{n} \exp\left(2i\pi \sum_{j=1}^{r} a_{h_j}\right), \qquad r \geqslant 1, \tag{6}$$

$$\zeta_0 = 1, \qquad \zeta_r = \sum_{1\leqslant h_1<h_2<\ \ <h_{r-1}\leqslant h_r}^{q} \exp\left(2i\pi \sum_{j=1}^{r} b_{h_j}\right), \qquad r \geqslant 1 \tag{7}$$

Then

$$\sum_{r=0}^{q-m} (-)^r \zeta_r x^r = \left(\sum_{r=0}^{n} (-)^r \xi_r x^r\right)\left(\sum_{k=0}^{\infty} \Omega(k)\, x^k\right), \tag{8}$$

and so with $\Omega(0) = 1$, we have the recursion formula

$$\Omega(k) = (-)^k(\zeta_k - \xi_k) + \sum_{r=1}^{k-1} (-)^{r-1} \xi_r \Omega(k-r), \qquad k \geqslant 1 \tag{9}$$

In particular,

$$\Omega(1) = \xi_1 - \zeta_1 \qquad \Omega(2) = \zeta_2 - \xi_2 + \xi_1\Omega(1),$$
$$\Omega(3) = \xi_2 - \zeta_3 + \xi_1\Omega(2) - \xi_2\Omega(1) \tag{10}$$

For an alternative recurrence formula, we proceed as follows Let Ω stand for the left-hand side of (5) The logarithmic derivative of Ω gives

$$D\Omega = \Omega\left[-\sum_{j=m+1}^{q} \frac{\exp(2i\pi b_j)}{1 - x\exp(2i\pi b_j)} + \sum_{j=1}^{n} \frac{\exp(2i\pi a_j)}{1 - x\exp(2i\pi a_j)}\right], \qquad D = \frac{d}{dx}$$

and by Leibnitz's rule

$$D^{r+1}\Omega = \sum_{k=0}^{r} \binom{r}{k} k!\,(D^{r-k}\Omega)$$

$$\times \left[-\sum_{j=m+1}^{q} \frac{\exp[2i\pi b_j(k+1)]}{(1 - x\exp[2i\pi b_j])^{k+1}} + \sum_{j=1}^{n} \frac{\exp[2i\pi a_j(k+1)]}{(1 - x\exp[2i\pi a_j])^{k+1}} \right].$$

Since $D^k\Omega$ evaluated at $x = 0$ is $k!\,\Omega(k)$, we find that

$$\Omega(k+1) = [1/(k+1)] \sum_{r=0}^{k} (S_{r+1} - T_{r+1})\,\Omega(k-r), \tag{11}$$

where

$$S_r = \sum_{j=1}^{n} \exp(2i\pi a_j r), \qquad T_r = \sum_{j=m+1}^{q} \exp(2i\pi b_j r), \qquad r \geqslant 1. \tag{12}$$

Observe that the ξ_r's and the S_r's are related by Newton's identities for symmetric functions, see Bochér (1936). Similar remarks pertain for the ζ_r's and the T_r's.

Let

$$\frac{\prod_{j=n+1}^{p} [1 - x\exp(2i\pi a_j)]}{\prod_{j=1}^{m} [1 - x\exp(2i\pi b_j)]} = \sum_{\lambda=0}^{\infty} E_p^{m,n}(\lambda)\, x^\lambda,$$

$$|\, x\exp(2i\pi b_j\,| < 1, \qquad E_p^{m,n}(\lambda) = E(\lambda), \tag{13}$$

$$\frac{\prod_{j=n+1}^{p} [1 - x\exp(2i\pi a_j)]}{[1 - x\exp(2i\pi a_l)] \prod_{j=1}^{m} [1 - x\exp(2i\pi b_j)]} = \sum_{r=0}^{\infty} \theta_p^{m,n}(l, r)\, x^r,$$

$$|\, x\exp(2i\pi b_j)| < 1, \qquad |\, x\exp(2i\pi a_l)| < 1,$$

$$\theta_p^{m,n}(l, r) = \theta(l, r). \tag{14}$$

By appropriate change of notation, recursion formulas for the evaluation of $E(\lambda)$ and $\theta(l, r)$ follow from (6)–(12).

Let

$$\Phi_{p,q}^{m,n}(h, \lambda) = \sum_{r=0}^{h+\lambda-1} E(r)\,\Omega^{0,p}_{q}(h + \lambda - r - 1), \qquad \Phi_{p,q}^{m,n}(h, \lambda) = \Phi(h, \lambda), \tag{15}$$

$$\psi_{p,q}^{m,n}(h, \lambda) = \sum_{r=1}^{h} E(\lambda - r)\,\tilde{\Omega}^{0,p}_{q}(h - r), \qquad \psi_{p,q}^{m,n}(h, \lambda) = \psi(h, \lambda), \tag{16}$$

$$R_{p\ q}^{m,n}(h,\lambda) = A^m\ {}_q^n\Phi(h,\lambda) - A^0\ {}_q^p\bar{B}\bar{\psi}(h,-\tau-\lambda+1), \qquad R_{p\ q}^{m\ n}(h,\lambda) = R(h,\lambda), \tag{17}$$

$$T_{p\ q}^{m\ n}(l\ \lambda) = -\{\exp(i\pi a_l)\,B\theta(l,\lambda-1) + \exp(-i\pi a_l)\,\bar{B}\bar{\theta}(l,-\tau-\lambda)\}$$
$$\times\ \Delta^0\ {}_q^p(l) + \exp[i\pi a_l(\tau+2\lambda-1)]\,\Delta^m\ {}_q^n(l)$$
$$T_{p\ q}^{m\ n}(l,\lambda) = T(l,\lambda) \tag{18}$$

$$D_{p\ q}^{m\ n}(\lambda) = (-2\pi i)^{p-q}\exp\left\{i\pi\left(\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right)\right\}$$
$$\times\ \{BE(\lambda) - \bar{B}\bar{E}(-\tau-\lambda)\}, \qquad D_{p\ q}^{m\ n}(\lambda) = D(\lambda) \tag{19}$$

5 9 3 Expansion Theorems

Theorem 1.1. *Under conditions* (A), (B), $1 \leqslant n \leqslant p \leqslant q$, $1 \leqslant m \leqslant q$, $\nu + 1 \leqslant 0$, λ *an arbitrary integer such that* $0 \leqslant \lambda \leqslant -\nu - 1$, *then*

$$G_{p\ q}^{m\ n}(z) = \sum_{t=1}^{n} \exp[-i\pi a_t(\nu+2\lambda+1)]\,\Delta(t)\,G_{p\ q}^{q\ 1}(z\exp[i\pi(\nu+2\lambda+1)]\,|\,a_t) \tag{1}$$

Theorem 1.2 *Under conditions* (A), (B), $1 \leqslant n \leqslant p < q$, $2 \leqslant m \leqslant q$, $\nu + 1 \leqslant 0$, $|\arg z| < \rho\pi$, λ *as in* (1), $-\eta\pi < \arg z + 2\lambda\pi < \rho\pi$, *then* (1) *expresses* $G_{p\ q}^{m\ n}(z)$ *in terms of fundamental solutions valid near* $z = \infty$ (2)

Equation (1) is readily proved from 5 7(6) and 5 2(11, 15) The special case $\lambda = 0$ of (1) is the statement 5 2(19) For the proof of (2), the inequality $-\eta\pi < \arg z + 2\lambda\pi < \rho\pi$ implies that $|\arg z + (\nu+1+2\lambda)\pi| < (\frac{1}{2}\sigma + 1)\pi$ whence 5 8(4) with λ replaced by $-\lambda$ holds for the functions $G_{p\ q}^{q\ 1}$ on the right of (1) Thus these functions are fundamental solutions

Theorem 2 1. *Under conditions* (A), (B), $q \geqslant 1$, $0 \leqslant n \leqslant p \leqslant q$, $0 \leqslant m \leqslant q$, r *an arbitrary integer such that* $r \geqslant \max(0, \nu+1)$, *then*

$$G_{p\ q}^{m\ n}(z) = A\sum_{k=0}^{r-1}\Omega(k)\,G_{p\ q}^{q\ 0}(z\exp[i\pi(\nu-2k)])$$
$$+ \sum_{t=1}^{n}\exp[i\pi a_t(2r-\nu-1)]\,\Delta(t)\,G_{p\ q}^{q\ 1}(z\exp[i\pi(\nu-2r+1)]\,|\,a_t) \tag{3}$$

This expansion is also valid if z is replaced by $-z$

Theorem 2.2. *Under conditions* (A), (B), $1 \leqslant n \leqslant p < q$, $2 \leqslant m \leqslant q$, $\nu + 1 \leqslant 0$, $-\eta\pi < \arg z < (\tau + \epsilon)\pi$, r *an arbitrary integer such that* $r \geqslant 0$ *and* $-\rho\pi < 2r\pi - \arg z < \eta\pi$, *then* (3) *expresses* $G^{m,n}_{p,q}(z)$ *in terms of fundamental solutions valid near* $z = \infty$. (4)

Theorem 2.3. *Under conditions* (A), (B), $0 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, $\frac{3}{4}p + \frac{1}{4}q - \frac{1}{2}\epsilon < m + n \leqslant q + 1$, $-\rho\pi < \arg z < (\tau + \epsilon)\pi$, r *an arbitrary integer such that* $r \geqslant \nu + 1$, $-\rho\pi < 2r\pi - \arg z < \eta\pi$, *then* (3) *expresses* $G^{m,n}_{p,q}(z)$ *in terms of fundamental solutions valid near* $z = \infty$. (5)

Theorem 2.4. *Under conditions* (A), (B), $1 \leqslant n \leqslant p < q$, $2 \leqslant m \leqslant q$, $\nu + 1 \leqslant 0$, $-(\tau + \epsilon)\pi < \arg z < -\eta\pi$, r *an arbitrary integer* $\geqslant 0$ *such that* $-\rho\pi < 2r\pi + \arg z < \eta\pi$, *then* (3) *with* i *replaced by* $-i$ *expresses* $G^{m,n}_{p,q}(z)$ *in terms of fundamental solutions valid near* $z = \infty$. (6)

Theorem 2.5. *Under conditions* (A), (B), $0 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, $\frac{3}{4}p + \frac{1}{4}q - \frac{1}{2}\epsilon < m + n \leqslant q + 1$, $-(\tau + \epsilon)\pi < \arg z < \rho\pi$, r *an arbitrary integer such that* $r \geqslant \nu + 1$, $-\rho\pi < 2r\pi + \arg z < \eta\pi$, *then* (3) *with* i *replaced by* $-i$ *expresses* $G^{m,n}_{p,q}(z)$ *in terms of fundamental solutions valid near* $z = \infty$. (7)

PROOF. First we take up proof of (3) for which we distinguish three cases. They are:

CASE 1. $1 \leqslant n \leqslant p \leqslant q, \nu + 1 \leqslant 0, r \geqslant 0.$

CASE 2. $1 \leqslant n \leqslant p \leqslant q, 0 \leqslant \nu + 1, r \geqslant \nu + 1.$

CASE 3. $n = 0, q \geqslant 1, 0 \leqslant p \leqslant q, 0 \leqslant m \leqslant q, r \geqslant 1 + q - m.$

We prove Case 1 only, as proof for the other cases is similar. The proof is by induction on r. Clearly when $r = 0$, (3) becomes (1). Let us assume that (3) is true with r replaced by $r - 1$. From 5.4(11),

$$G^{q,1}_{p,q}(z \exp[i\pi(\nu - 2r + 3)]|| a_t) = \exp(2\pi i a_t)\, G^{q,1}_{p,q}(z \exp[i\pi(\nu - 2r + 1)]|| a_t)$$
$$- 2\pi i \exp(i\pi a_t)\, G^{q,0}_{p,q}(z \exp[i\pi(\nu - 2r + 2)]).$$

Put the latter into $\sum_{t=1}^{n}$ of (3) with r replaced by $r - 1$. Then we get the sum $\sum_{t=1}^{n}$ in (3) and also the sum

$$-2\pi i\, G^{q,0}_{p,q}(z \exp[i\pi(\nu - 2r + 2)]) \sum_{t=1}^{n} \exp[-i\pi a_t(\nu - 2r + 2)]\, \Delta(t).$$

It may be shown from the definitions in 5 9 2 that

$$\sum_{i=1}^{n} \exp[-i\pi a_i(\nu - 2r - 2)]\, \Delta(t) = -(1/2\pi i)[A\Omega(r-1) - \bar{A}\bar{\Omega}(\nu + 1 - r)],$$

and in the present instance this reduces to

$$-(1/2\pi i)\, A\Omega(r-1)$$

since $\bar{\Omega}(\nu + 1 - r)$ vanishes when $\nu + 1 - r \leqslant -1$ Thus the sum $\sum_{k=0}^{r-2}$ on the right-hand side of (3) [recall we assumed (3) true with r replaced by $r - 1$] becomes, because of the above substitution, the sum $\sum_{k=0}^{r-1}$ of (3) and proof of (3) for Case 1 is completed

Next we turn to the proof of (4) After the manner of the proof of (2), it is clear that the $G_{p\ q}^{q\ 1}$ functions on the right-hand side of (3) are fundamental solutions From the inequalities involving $\arg z$ in (4), $4r\pi < (3q - 3p + 2\epsilon + 4)$ which implies that $r < q - p + 1$ Hence, the number of functions $G_{p\ q}^{q\ 0}$ on the right-hand side of (3) is at most $q - p$ These functions satisfy 5 8(5) with $\psi = r$ and so are fundamental solutions The proof of (5)–(7) is similar and we omit details

Theorem 3 1. *Under conditions* (A), (B), $q \geqslant 1$, $0 \leqslant n \leqslant p \leqslant q$, $0 \leqslant m \leqslant q$, $\nu + 1 \geqslant 0$, *r an arbitrary integer such that* $0 \leqslant r \leqslant \nu + 1$, *then*

$$G_{p\ q}^{m\ n}(z) = A \sum_{k=0}^{r-1} \Omega(k)\, G_{p\ q}^{q\ 0}(ze^{i\pi(\nu-2k)}) + \bar{A} \sum_{k=0}^{\nu-r} \bar{\Omega}(k)\, G_{p\ q}^{q\ 0}(ze^{-i\pi(\nu-2k)})$$

$$+ \sum_{i=1}^{n} \exp[i\pi a_i(2r - \nu - 1)]\, \Delta(t)\, G_{p\ q}^{q\ 1}(z \exp[i\pi(\nu - 2r + 1)] \mid a_i) \qquad (8)$$

This expansion formula is also valid if z is replaced by $-z$ provided that $r \geqslant \max(0, \nu + 1)$

Theorem 3.2 *Under conditions* (A), (B), $0 \leqslant n \leqslant p < q$ $1 \leqslant m \leqslant q$, $p + 1 \leqslant m + n \leqslant \frac{3}{4}q + \frac{1}{4}p - \frac{1}{2}\epsilon + 1$, $|\arg z| < (\tau + \epsilon)\pi$, *$r$ an arbitrary integer such that* $0 \leqslant r \leqslant \nu + 1$, $-\rho\pi < 2r\pi - \arg z < \eta\pi$, *then* (8) *expresses* $G_{p\ q}^{m\ n}(z)$ *in terms of fundamental solutions valid near* $z = \infty$ (9)

Theorem 4.1. *Under conditions* (A), (C), $q \geqslant 1$, $0 \leqslant n \leqslant p \leqslant q$, $0 \leqslant m \leqslant q$, *$\lambda$ and μ arbitrary integers with* $0 \leqslant \mu \leqslant q - p$, *then*

$$G_{p\ q}^{m\ n}(z) = \sum_{h=1}^{q-p-\mu} R(h, \lambda)\, G_{p\ q}^{q\ 0}(z \exp[i\pi(\nu - 2h - 2\lambda + 2)])$$

$$+ \sum_{k=1}^{\mu} R(k, -\tau - \lambda + 1)\, G_{p,q}^{q,0}(z \exp[i\pi(2p - q - m - n + 2k - 2\lambda)])$$

$$+ \sum_{k=1}^{p} \exp[i\pi a_k(q - p - 2\mu)]\, T(k, \lambda)$$

$$\times\, G_{p,q}^{q,1}(z \exp[i\pi(2p - q - m - n - 2\lambda + 2\mu + 1)] \| a_k). \tag{10}$$

Theorem 4.2. *Under conditions* (A), (C), $0 \leqslant n \leqslant p < q, 0 \leqslant m \leqslant q$, λ *and* μ *arbitrary integers such that* $(\tau + \epsilon + 2\lambda - 2)\pi \leqslant \arg z <$ $(\tau + \epsilon + 2\lambda)\pi$, $(m + n - \frac{3}{2}p + \frac{1}{2}q + 2\lambda - 2)\pi < 2\mu\pi + \arg z <$ $(m + n - \frac{5}{2}p + \frac{3}{2}q + 2\lambda)\pi$, *then* (10) *expresses* $G_{p,q}^{m,n}(z)$ *in terms of fundamental solutions valid near* $z = \infty$. (11)

Proof of (8)–(11) is much like that for the previous theorems. We omit details and refer the reader to the papers by Meijer (1946).

5.10. Asymptotic Expansions of $G_{p,q}^{m,n}(z)$

As previously noted, asymptotic expansions of $G_{p,q}^{m,n}(z)$ for all values of m, n, p, q (with $p < q$), and $\arg z$ for $|z| \to \infty$ can be derived from the theorems of 5.9.3 and Theorems 1 and 5 of 5.7. Indeed, in his work Meijer (1946) investigates all of the G-functions which appear on the right-hand side of the expansion formulas in 5.9.3, and from these he determines those which are dominant.

It is helpful to clarify the concept of dominance. We say that $A(z)$ is dominant compared with $B(z)$ or $B(z)$ is subdominant with respect to $A(z)$ if the order of the lead term of the asymptotic expansion of $B(z)$ is less than the order of the error term in the asymptotic expansion of $A(z)$. To illustrate, consider the asymptotic expansions

$$A_1(z) \sim e^z \sum_{k=0}^{\infty} a_{1k} z^{-k}, \qquad A_2(z) \sim z^6 \sum_{k=0}^{\infty} a_{2k} z^{-k},$$

$$A_3(z) \sim z^{1/2} \sum_{k=0}^{\infty} a_{3k} z^{-k}, \qquad A_4(z) \sim e^{iz} z^{-8} \sum_{k=0}^{\infty} a_{4k} z^{-k},$$

$$A_5(z) \sim e^{-z} \sum_{k=0}^{\infty} a_{5k} z^{-k}, \qquad A_6(z) \sim e^{-2z} \sum_{k=0}^{\infty} a_{6k} z^{-k},$$

where it is supposed that no a_{i0}, $i = 1, 2, \ldots, 6$, is zero. Assume $z > 0$. Then clearly $A_1(z)$ is dominant compared with $A_2(z), \ldots, A_6(z)$. Compared

with $A_5(z)$ and $A_6(z)$, $A_2(z)$, $A_3(z)$, and $A_4(z)$ are dominant, and $A_5(z)$ is dominant compared with $A_6(z)$ However, $A_2(z)$ is not dominant compared with $A_3(z)$ and $A_4(z)$ Thus among the functions $A_2(z)$, , $A_6(z)$, there are three dominants They are $A_2(z)$, $A_3(z)$, and $A_4(z)$ Qualitatively, we say that $A_1(z)$ is an exponential growth, $A_5(z)$ and $A_6(z)$ are exponential decays, $A_2(z)$ and $A_3(z)$ are algebraic, and $A_4(z)$ is algebraic and sinusoidal

To get asymptotic expansions for $G_{p\,q}^{m\,n}(z)$ it is only necessary to retain the dominant term or terms in the right-hand sides of the expansion formulas in 5 9 3, unless, of course, the coefficients of these dominants vanish The coefficients depend on the parameters a_j and b_j and in general do not vanish We suppose, therefore, that if there is only one dominant function, the coefficient of this function is not zero If there are two or more dominants, then at least one of them has a nonvanishing coefficient If all the coefficients of the dominant functions vanish, it is necessary to further explore the nature of the above expansion formulas We omit such details

We follow Meijer and give only the dominant term or terms in the asymptotic expansions However, for the benefit of the reader, we tell from which expansion theorems the results follow so that the complete expansion may be deduced if desired This is illustrated later when we record the complete asymptotic expansion for the ${}_pF_q$, $p \leqslant q$ In the theorems which follow, we make free use of the notation introduced in 5 7 and 5 9 2

Theorem 1. *Under conditions* (A), (B), $1 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, $\rho > 0$, *then*

$$G_{p\,q}^{m\,n}(z) \sim \sum_{t=1}^{n} \exp[-i\pi a_t(\nu+1)]\, \Delta(t)\, E_{p\,q}(z \exp[i\pi(\nu+1)] \mid a_t),$$

$$|z| \to \infty, \qquad |\arg z| \leqslant \rho\pi - \delta \qquad \delta > 0 \tag{1}$$

This is proved from 5 9 3(1, 2) if $\nu + 1 \leqslant 0$ and from 5 9 3(3, 5) if $\rho > 0$ and $\nu + 1 \geqslant 0$

Theorem 2 *If* $1 \leqslant p + 1 \leqslant m \leqslant q$, *then*

$$G_{p\,q}^{m\,0}(z) \sim A^m\, {}_qH_{p\,q}(ze^{i\pi(q-m)}) \qquad \text{if} \quad m \leqslant q - 1,$$

$$|z| \to \infty, \qquad \delta \leqslant \arg z \leqslant (m - p + 1)\pi - \delta, \qquad \delta > 0, \tag{2}$$

$$G_{p\,q}^{m\,0}(z) \sim \bar{A}^m\, {}_qH_{p\,q}(ze^{-i\pi(q-m)}) \qquad \text{if} \quad m \leqslant q - 1$$

$$|z| \to \infty, \qquad \delta - (m - p + 1)\pi \leqslant \arg z \leqslant -\delta, \qquad \delta > 0, \tag{3}$$

$$G_{p,q}^{m,0}(z) \sim A^{m,0}{}_{q}H_{p,q}(ze^{i\pi(q-m)}) + \bar{A}^{m,0}{}_{q}H_{p,q}(ze^{-i\pi(q-m)}),$$
$$\text{if} \quad m \leqslant (q-1), \quad z \to +\infty, \qquad \text{that is} \quad \arg z = 0; \tag{4}$$

$$G_{p,q}^{q,0}(z) \sim H_{p,q}(z),$$
$$|z| \to \infty, \qquad |\arg z| \leqslant (\sigma + \epsilon)\pi - \delta, \qquad \delta > 0. \tag{5}$$

Equations (2)–(4) are proved from 5.9.3(3). Equation (5) is the same as 5.7(12).

Theorem 3. *Under conditions* (A), $0 \leqslant n \leqslant p \leqslant q-2$, $1 \leqslant \tau \leqslant \frac{1}{2}\sigma$ [*or* $p+1 \leqslant m+n \leqslant \frac{1}{2}(p+q)$], *then*

$$G_{p,q}^{m,n}(z) \sim AH_{p,q}(ze^{i\pi\nu}),$$
$$|z| \to \infty, \qquad \delta \leqslant \arg z \leqslant (\tau+1)\pi - \delta, \qquad \delta > 0; \tag{6}$$

$$G_{p,q}^{m,n}(z) \sim \bar{A}H_{p,q}(ze^{-i\pi\nu}),$$
$$|z| \to \infty, \qquad \delta - (\tau+1)\pi \leqslant \arg z \leqslant -\delta, \qquad \delta > 0; \tag{7}$$

$$G_{p,q}^{m,n}(z) \sim AH_{p,q}(ze^{i\pi\nu}) + \bar{A}H_{p,q}(ze^{-i\pi\nu}) \qquad \text{with} \quad 1 \leqslant \tau < \tfrac{1}{2}\sigma,$$
$$z \to +\infty, \qquad \text{that is} \quad \arg z = 0; \tag{8}$$

$$G_{p,q}^{m,n}(z) \sim AH_{p,q}(ze^{i\pi\nu}) + \bar{A}H_{p,q}(ze^{-i\pi\nu})$$
$$+ \sum_{t=1}^{n} \exp[-i\pi a_t(\nu+1)]\, \Delta(t)\, E_{p,q}(ze^{i\pi(\nu+1)} \,\|\, a_t) \tag{9}$$

with $\rho = 0$, *condition* (B) *and* $z \to +\infty$, *that is*, $\arg z = 0$.

This is proved from 5.9.3(8, 9).

Theorem 4. *Under conditions* (A), $0 \leqslant n \leqslant p < q$, $1 \leqslant m \leqslant q$, $\rho > 0$, *then*

$$G_{p,q}^{m,n}(z) \sim AH_{p,q}(ze^{i\pi\nu}),$$
$$|z| \to \infty, \qquad \delta + \rho\pi \leqslant \arg z \leqslant (\tau+\epsilon)\pi - \delta, \qquad \delta > 0; \tag{10}$$

$$G_{p,q}^{m,n}(z) \sim \bar{A}H_{p,q}(ze^{-i\pi\nu}),$$
$$|z| \to \infty, \qquad \delta - (\tau+\epsilon)\pi < \arg z \leqslant -\rho\pi + \delta, \qquad \delta > 0; \tag{11}$$

$$G_{p,q}^{m,n}(z) \sim AH_{p,q}(ze^{i\pi\nu}) + \sum_{t=1}^{n} \exp[-i\pi a_t(\nu+1)]\, \Delta(t)\, E_{p,q}(ze^{i\pi(\nu+1)} \,\|\, a_t), \tag{12}$$

provided also that condition (B) *holds*, $|z| \to \infty$, $\arg z = \rho\pi$,

$$G_{p\,q}^{m\,n}(z) \sim \bar{A} H_{p\,q}(ze^{-i\pi\nu}) + \sum_{t=1}^{n} \exp[i\pi a_t(\nu+1)]\, \Delta(t)\, E_{p\,q}(ze^{-i\pi(\nu+1)} \| a_t), \tag{13}$$

provided also that condition (B) *holds*, $|z| \to \infty$, $\arg z = -\rho\pi$

Equations (10) and (12) follow from 5 9 3(3, 4, 5), respectively, while (11) and (13) follow from 5 9 3(3, 6, 7), respectively

Theorem 5. *Under conditions* (A), $0 \leqslant n \leqslant p \leqslant q-2$, $1 \leqslant m \leqslant q$, *if* $\tau \leqslant 1$, λ *is an arbitrary integer, if* $\tau \geqslant 2$, λ *is either an arbitrary integer* $\geqslant 0$ *or an arbitrary integer* $\leqslant -\tau$, *then*

$$G_{p\,q}^{m\,n}(z) \sim D(\lambda)\, H_{p\,q}(ze^{i\pi(\nu-2\lambda)}),$$

$$|z| \to \infty, \qquad \delta_1 + (\tau + 2\lambda - 1)\pi \leqslant \arg z \leqslant (\tau + 2\lambda + 1)\pi + \delta_2, \tag{14}$$

where δ_1 *and* δ_2 *are arbitrary small quantities whose signs are chosen so that the closed interval* $[\delta_1 + a, \delta_2 + b]$ *is contained in the open interval* (a, b),

$$G_{p\,q}^{m\,n}(z) \sim D(\lambda)\, H_{p\,q}(z \exp[i\pi(\nu - 2\lambda)]) + D(\lambda - 1)\, H_{p\,q}(z \exp[i\pi(\nu - 2\lambda + 2)]),$$

$$q > p + 2, \qquad |z| \to \infty, \qquad \arg z = (\tau + 2\lambda - 1)\pi, \tag{15}$$

$$\begin{aligned} G_{p\,p+2}^{m\,n}(z) \sim{}& D_{p\,p+2}^{m\,n}(\lambda)\, H_{p\,p+2}(z \exp[-i\pi(\tau + 2\lambda - 2)]) \\ &+ D_{p\,p+2}^{m\,n}(\lambda - 1)\, H_{p\,p+2}(z \exp[-i\pi(\tau + 2\lambda - 4)]) \\ &+ \sum_{t=1}^{p} \exp[-2i\pi(\lambda + 1)\, a_t]\, T_{p\,p+2}^{m\,n}(t, \lambda)\, E_{p\,p+2}(z \exp[-i\pi(\tau - 3)] \| a_t), \end{aligned} \tag{16}$$

provided condition (C) *holds*, $|z| \to \infty$, $\arg z = (\tau + 2\lambda - 1)\pi$

This is proved from 5 9 3(10, 11) Note that the case $m = 0$ need not be considered in view of 5 2(8) For special values of τ and λ, Theorem 5 reduces to previous statements These are enumerated as follows

If $\tau \geqslant 1$ and $\lambda = 0$, then (14) reduces to (6)
If $\tau \geqslant 1$ and $\lambda = -\tau$, then (14) reduces to (7)
If $\tau > 1$ and $\lambda = 0$, then (15) reduces to (6)
If $\tau = 1$ and $\lambda = 0$, then (15) reduces to (8) with $\tau = 1$
If $\tau > 1$ and $\lambda = 0$, then (16) reduces to (12) with $q = p + 2$
If $\tau = 1$ and $\lambda = 0$, then (16) reduces to (9) with $q = p + 2$

Theorem 6.1. *Under the conditions* (A), (C), $p \geqslant 1$, $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant p+1$, λ *an arbitrary integer, then*

$$G_{p,p+1}^{m,n}(z) \sim \sum_{t=1}^{p} \exp[-i\pi a_t(2\lambda+1)]\, T_{p,p+1}^{m,n}(t,\lambda)\, E_{p,p+1}(z \exp[-i\pi(\tau-2)] \| a_t),$$

$$|z| \to \infty, \qquad \delta_1 + (\tau + 2\lambda - \tfrac{3}{2})\pi \leqslant \arg z \leqslant (\tau + 2\lambda - \tfrac{1}{2})\pi + \delta_2. \tag{17}$$

This expansion is not valid when $n = \lambda = 0$ and $m = p+1$. For $n = 0$ and $m = p+1$, see (2)–(5). The expansion (17) is also valid if

$$\tau \geqslant 2, \qquad -\tau < \lambda < 0, \qquad (\tau + 2\lambda - \tfrac{1}{2})\pi \leqslant \arg z \leqslant (\tau + 2\lambda + \tfrac{1}{2})\pi + \delta_2$$

or

$$\tau \geqslant 2, \qquad 1 - \tau < \lambda < 1, \qquad \arg z = (\tau + 2\lambda - \tfrac{3}{2})\pi.$$

REMARK. If $1 \leqslant t \leqslant n$ and $1 - \tau \leqslant \lambda \leqslant 0$, then $T_{p,p+1}^{m,n}(t,\lambda) = \exp[i\pi a_t(\tau + 2\lambda - 1)]\, \Delta_{p+1}^{m,n}(t)$, and if $n+1 \leqslant t \leqslant p$ and $1 - \tau \leqslant \lambda \leqslant 0$, then $T_{p,p+1}^{m,n}(t,\lambda) = 0$, so that (17) with $n \geqslant 1$, $\tau \geqslant 1$, and $1 - \tau \leqslant \lambda \leqslant 0$ is the same as (1) with $q = p+1$.

Theorem 6.2. *Under the conditions* (A), $p \geqslant 1$, $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant p+1$, $\tau \geqslant 2$, λ *an arbitrary integer* $\geqslant 0$ *or* $\leqslant -\tau$,

$$G_{p,p+1}^{m,n}(z) \sim D_{p,p+1}^{m,n}(\lambda)\, H_{p,p+1}(z \exp[-i\pi(\tau + 2\lambda - 1)]),$$

$$\delta_1 + (\tau + 2\lambda - \tfrac{1}{2})\pi \leqslant \arg z \leqslant (\tau + 2\lambda + \tfrac{1}{2})\pi + \delta_2. \tag{18}$$

This is also valid if $\tau \leqslant 1$ *with* λ *an arbitrary integer.*

Theorem 6.3. *Under conditions* (A), (C), $p \geqslant 1$, $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant p+1$, $\tau \geqslant 2$, λ *an arbitrary integer* $\geqslant 0$ *or* $\leqslant -\tau$,

$$G_{p,p+1}^{m,n}(z) \sim D_{p,p+1}^{m,n}(\lambda)\, H_{p,p+1}(z \exp[-i\pi(\tau + 2\lambda - 1)])$$

$$+ \sum_{t=1}^{p} \exp[-i\pi a_t(2\lambda+1)]\, T_{p,p+1}^{m,n}(t,\lambda)\, E_{p,p+1}(z \exp[-i\pi(\tau-2)] \| a_t),$$

$$|z| \to \infty, \qquad \arg z = (\tau + 2\lambda - \tfrac{1}{2})\pi. \tag{19}$$

This is also valid if $\tau \leqslant 1$ *and* λ *is an arbitrary integer.*

Theorem 6 4. *Under conditions* (A), (C), $p \geqslant 1$, $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant p+1$, $\tau \geqslant 2$, λ *an arbitrary integer* $\geqslant 1$ *or* $\leqslant -\tau+1$,

$$G_{p\ p+1}^{m\ n}(z) \sim D_{p\ p+1}^{m\ n}(\lambda-1)\, H_{p\ p+1}(z \exp[-i\pi(\tau+2\lambda-3)])$$

$$+ \sum_{t=1}^{p} \exp[-i\pi a_t(2\lambda+1)]\, T_{p\ p+1}^{m\ n}(t,\lambda)\, E_{p\ p+1}(z \exp[-i\pi(\tau-2)] \| a_t),$$

$$|z| \to \infty, \qquad \arg z = (\tau+2\lambda-\tfrac{3}{2})\pi \tag{20}$$

This is also valid if $\tau \leqslant 1$ *and* λ *is an arbitrary integer*

Equations (17)–(20) are proved from 5 9 3(10, 11)

The following and last theorem of this section concerns the analytic continuation of $G_{p\ p}^{m\ n}(z)$ in the general case

Theorem 7. *Under conditions* (A), (C), $p \geqslant 1$, $0 \leqslant n \leqslant p$, $0 \leqslant m \leqslant p$, λ *an arbitrary integer such that* $(\tau+2\lambda-2)\pi < \arg z < (\tau+2\lambda)\pi$, *then* $G_{p\ p}^{m\ n}(z)$ *can be expressed in terms of fundamental solutions valid near* $z = \infty$ *by the formula*

$$G_{p\ p}^{m\ n}(z) = \sum_{t=1}^{p} T_{p\ p}^{m\ n}(t,\lambda)\, G_{p\ p}^{p\ 1}(z \exp[-i\pi(\tau+2\lambda-1)] \| a_t), \tag{21}$$

and can be continued analytically outside the unit circle $|z| = 1$ *by the expansion*

$$G_{p\ p}^{m\ n}(z) = \sum_{t=1}^{p} \exp(-2i\pi\lambda a_t)\, T_{p\ p}^{m\ n}(t,\lambda)\, E_{p\ p}(z \exp[-i\pi(\tau-1)] \| a_t) \tag{22}$$

If $\tau = m+n-p \geqslant 1$ *and* $1-\tau \leqslant \lambda \leqslant 0$, *and we use the properties of* $T_{p\ p}^{m\ n}(t,\lambda)$ *as in the remark following* (17), *then*

$$G_{p\ p}^{m\ n}(z) = \sum_{t=1}^{n} \exp[i\pi a_t(\tau-1)]\, \Delta^{m\ n}_{p}(t)\, E_{p\ p}(z \exp[-i\pi(\tau-1)] \| a_t)$$

$$|\arg z| < \tau\pi, \qquad \arg z \neq (\tau-2k)\pi, \qquad k = 1, 2, \ldots, [\tau/2] \tag{23}$$

Observe that if $p = q$, $-\tau = \nu = \rho$ *and the expansion in* (23) *is the same as that in* (1) *with* $p = q$

Theorem 7 follows from 5 9 3(10, 11) It is essentially the same as the Theorem given by 5 7(10) though slightly less general for we have now excluded those values of z for which $\arg z = (\tau-2k)\pi$, $k = 1, 2, \ldots, [\tau/2]$

5.11. Asymptotic Expansions of ${}_pF_q(z)$ for Large z

5.11.1. Preliminary Results

From 5.2(14), we have

$$ {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| -z\right) = \frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)} G^{1,p}_{p,q+1}\left(z\middle|\begin{matrix}1-\alpha_p\\ 1-\rho_q\end{matrix}\right), \tag{1}$$

$$ {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) = \frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)} G^{1,p}_{p,q+1}\left(ze^{i\pi}\middle|\begin{matrix}1-\alpha_p\\ 1-\rho_q\end{matrix}\right), \tag{2}$$

where there are now $(q+1)$ ρ_h's and $\rho_0 = 1$. In 5.11.2 we give not only the dominant terms in the asymptotic expansion for ${}_pF_q(z)$ when $p < q$, but also the complete asymptotic expansion. These follow from 5.10(6–9) and 5.9.3(8, 9), respectively. In 5.11.3 we present the complete asymptotic expansion for ${}_pF_p(z)$ which follows from 5.9.3(3–7). The dominant terms in this last instance may be found from 5.10(1, 10–13).

In the theorems referenced above, let us put

$$m = 1, \qquad n = p \quad \text{and replace} \quad q \quad \text{by} \quad q+1, \tag{3}$$

$$\begin{aligned} a_h &= 1-\alpha_h, & h &= 1, 2, \ldots, p,\\ b_1 &= 0, & b_h &= 1-\rho_{h-1}, \qquad h = 2, 3, \ldots, q+1. \end{aligned} \tag{4}$$

The parameter σ [see 5.7(1)] is now replaced by

$$\beta = q + 1 - p. \tag{5}$$

We suppose that none of the parameters α_h is a negative integer or zero [condition (A)] and that no two of the α_h's differ by an integer or zero [condition (B)]. The latter condition can be relaxed. See the remark after (8).

The sum

$$\sum_{t=1}^{p} \exp[-i\pi\beta(1-\alpha_t)]\, \Delta^{1,p}_{q+1}(t)\, E_{p,q+1}(ze^{-i\pi\beta} \,\|\, 1-\alpha_t) \tag{6}$$

equals the same sum with i replaced by $-i$. This readily follows from the definition 5.7(7). With the aid of 5.9.2(1), (6) may be expressed formally in the form

$$L_{p,q}(z) = \sum_{t=1}^{p} L^{(t)}_{p,q}(z),$$

$$L^{(t)}_{p,q}(z) = \frac{z^{-\alpha_t}\Gamma(\alpha_t)\,\Gamma(\alpha_p-\alpha_t)^*}{\Gamma(\rho_q-\alpha_t)}\, {}_{q+1}F_{p-1}\left(\begin{matrix}\alpha_t,\ 1+\alpha_t-\rho_q\\ 1+\alpha_t-\alpha_p^*\end{matrix}\middle| \frac{(-)^{q-p}}{z}\right), \tag{7}$$

or formally from 5 2(11) or 5 1(27) without the restriction on p, q, and z,

$$L_{p,q}(z) = G^{p\,1}_{q+1\,p}\left(z^{-1}\,\middle|\,\begin{matrix}1, \rho_q\\ \alpha_p\end{matrix}\right) \tag{8}$$

Observe that if two or more of the α_h's differ by an integer or zero, we can define $L_{p\,q}(z)$ by a limit process See the discussion surrounding 5 1(27–30)

Next we consider the sum

$$A^{1\,p}_{q+1} \sum_{k=0}^{r-1} \Omega^{1\,p}_{q+1}(k)\, H_{p\,q+1}(z \exp[i\pi(q-p-2k)]), \tag{9}$$

which is equal to the same sum with z replaced by $-z$ in view of 5 9 2(2, 5) and 5 7(13) See also the statement following 5 9 3(3) From 5 9 2(5), and using (4), we have

$$\frac{\prod_{j=1}^{q}\{1 - x\exp(2i\pi\rho_j)\}}{\prod_{j=1}^{p}\{1 - x\exp(-2i\pi\alpha_j)\}} = \sum_{k=0}^{\infty} \Gamma^{1\,p}_{q+1}(k)\, x^k, \qquad |x\exp(-2i\pi\alpha_j)| < 1, \tag{10}$$

and from 5 9 2(6–12), we easily recover corresponding results for $\Gamma^{1\,p}_{q+1}(k)$ of (10) Let

$$\begin{aligned} &\xi_0 = 1, \quad \xi_1 = \sum_{j=1}^{p} \exp(-2i\pi\alpha_j), \quad \xi_2 = \sum_{1\leqslant j<k\leqslant p} \exp[-2i\pi(\alpha_j+\alpha_k)], \quad , \\ &\xi_r = \sum_{1\leqslant h_1<h_2<\ \ h_{r-1}\leqslant h_r}^{p} \exp\left(-2i\pi \sum_{j=1}^{r} \alpha_{h_j}\right), \qquad r \geqslant 1, \end{aligned} \tag{11}$$

$$\zeta_0 = 1, \quad \zeta_r = \sum_{1\leqslant h_1<h_2<\ \ h_{r-1}\leqslant h_r}^{q} \exp\left(-2i\pi \sum_{j=1}^{r} \rho_{h_j}\right), \quad r \geqslant 1 \tag{12}$$

Then

$$\sum_{r=0}^{q} (-)^r \zeta_r x^r = \left(\sum_{r=0}^{p} (-)^r \xi_r x^r\right)\left(\sum_{k=0}^{\infty} \Gamma^{1\,p}_{q+1}(k)\, x^k\right), \tag{13}$$

and so with $\Gamma^{1\,p}_{q+1}(0) = 1$,

$$\Gamma^{1\,p}_{q+1}(k) = (-)^k(\zeta_k - \xi_k) + \sum_{r=1}^{k-1} (-)^{r-1}\xi_r \Gamma^{1\,p}_{q+1}(k-r), \qquad k \geqslant 1, \tag{14}$$

$$\begin{aligned} &\Gamma^{1\,p}_{q+1}(1) = \xi_1 - \zeta_1, \qquad \Gamma^{1\,p}_{q+1}(2) = \zeta_2 - \xi_2 + \xi_1\Gamma^{1\,p}_{q+1}(1), \\ &\Gamma^{1\,p}_{q+1}(3) = \xi_3 - \zeta_3 + \xi_1\Gamma^{1\,p}_{q+1}(2) - \xi_2\Gamma^{1\,p}_{q+1}(1) \end{aligned} \tag{15}$$

From 5.9.2(11, 12), we have

$$\Gamma^{1,p}_{q+1}(k+1) = \frac{1}{k+1}\sum_{r=0}^{k}(S_{r+1} - T_{r+1})\,\Gamma^{1,p}_{q+1}(k-r),$$

$$S_r = \sum_{j=1}^{p}\exp(-2i\pi\alpha_j r), \qquad T_r = \sum_{j=1}^{q}\exp(-2i\pi\rho_j r), \qquad r \geqslant 1, \tag{16}$$

$$\Gamma^{1,p}_{q+1}(2) = \sum_{\substack{j,k=1\\ j\neq k}}^{p}\exp[-2i\pi(\alpha_j+\alpha_k)] + \sum_{\substack{j,k=1\\ j\neq k}}^{q}\exp[-2i\pi(\rho_j+\rho_k)]$$

$$-\sum_{j=1}^{p}\sum_{k=1}^{q}\exp[-2i\pi(\alpha_j+\rho_k)]. \tag{17}$$

We return to (9). Now in our present notation,

$$A^{1,p}_{q+1}H_{p,q+1}(z\exp[i\pi(q-p-2k)]) = K_{p,q}(z\exp[-i\pi(2k+1)]), \tag{18}$$

$$K_{p,q}(z) = \frac{(2\pi)^{(1-\beta)/2}}{\beta^{1/2}}[\exp\{\beta z^{1/\beta}\}]\,z^{\gamma}\sum_{r=0}^{\infty}N_r z^{-r/\beta}, \qquad N_0 = 1,$$

$$\beta = q+1-p, \quad \beta\gamma = (\beta-1)/2 + B_1 - C_1, \quad B_1 = \sum_{h=1}^{p}\alpha_h, \quad C_1 = \sum_{h=1}^{q}\rho_h. \tag{19}$$

The N_r's can be deduced from the developments in 5.11.5. If $q = p+1$, see 5.11.5(19). For $r = 1$, from 5.7(15) and 5.11.5(1), we have

$$\begin{aligned} N_1 &= C_2 - B_2 + (2\beta)^{-1}(B_1 - C_1)[\beta(B_1+C_1) + B_1 - C_1 - 2] \\ &\quad + (24\beta)^{-1}(\beta-1)(\beta-11), \\ B_2 &= \sum_{s=2}^{p}\sum_{t=1}^{s-1}\alpha_s\alpha_t, \qquad C_2 = \sum_{s=2}^{q}\sum_{t=1}^{s-1}\rho_s\rho_t. \end{aligned} \tag{20}$$

For convenience in the applications, we record

$$K_{p,q}(ze^{i\pi}) + K_{p,q}(ze^{-i\pi}) = \frac{2(2\pi)^{(1-\beta)/2}}{\beta^{1/2}}[\exp\{\beta z^{1/\beta}\cos\pi/\beta\}]\,z^{\gamma}$$

$$\times\sum_{r=0}^{\infty}N_r z^{-r/\beta}\cos(\pi r/\beta - \pi\gamma - \beta z^{1/\beta}\sin\pi/\beta). \tag{21}$$

It is clear that the substitutions (4) transform (9) into

$$\sum_{k=0}^{r-1} \Gamma^{1}{}_{q+1}^{p}(k)\, K_{p,q}(z \exp[-i\pi(2k+1)]) \tag{22}$$

which is equal to the same sum with i replaced by $-i$ Notice that the expressions

$$A^{1}{}_{q+1}^{p} H_{p,q+1}(ze^{i\pi\beta}) \quad \text{and} \quad \bar{A}^{1}{}_{q+1}^{p} H_{p,q+1}(ze^{-i\pi\beta})$$

are equal and reduce to $K_{p,q}(z)$ when (4) is used

5 11 2 Asymptotic Expansion of ${}_pF_q(z)$ for Large z, $0 \leqslant p \leqslant q-1$

The asymptotic expansions for ${}_pF_q(z)$ are divided into two cases Here we consider the case $0 \leqslant p \leqslant q-1$, that is, $\beta \geqslant 2$ The case $p = q \geqslant 1$ is treated in 5 11 3 We omit proof, as this has been sufficiently detailed in 5 11 1 See this last section for all necessary notation Some further representations are given in 5 11 4

$$\begin{aligned} {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| -z\right) \sim \frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)} \Bigg\{ &\sum_{k=0}^{r-1} \Gamma^{1}{}_{q+1}^{p}(k)\, K_{p,q}(z \exp[-i\pi(2k+1)]) \\ &+ \sum_{s=0}^{\beta-r-1} \bar{\Gamma}^{1}{}_{q+1}^{p}(k)\, K_{p,q}(z \exp[i\pi(2s+1)]) + L_{p,q}(z) \Bigg\}, \\ &0 \leqslant p \leqslant q-1, \end{aligned} \tag{1}$$

where $|z| \to \infty$, r an arbitrary integer such that $0 \leqslant r \leqslant \beta$,

$$|\arg z| \leqslant 2\pi - \delta \qquad \delta > 0$$
$$\delta_1 + (4r - 3\beta - 2)\pi/2 \leqslant \arg z \leqslant (4r - \beta + 2)\pi/2 + \delta_2$$

where δ_1 and δ_2 have the same meaning as in 5 10(14), and $\bar{\Gamma}^{1}{}_{q+1}^{p}(k)$ is $\Gamma^{1}{}_{q+1}^{p}(k)$ with i replaced by $-i$ If we take $r = [(\beta+1)/2]$, then the restrictions on $\arg z$ are

$$\begin{aligned} \delta - 3\pi/2 \leqslant \arg z \leqslant 2\pi - \delta \qquad &\text{if} \quad \beta = 3 \\ |\arg z| \leqslant 2\pi - \delta \qquad &\text{if} \quad \beta \neq 3, \quad \delta > 0, \end{aligned} \tag{2}$$

while if $r = [\beta/2]$,

$$\begin{aligned} \delta - 2\pi \leqslant \arg z \leqslant 3\pi/2 - \delta \qquad &\text{if} \quad \beta = 3, \\ |\arg z| \leqslant 2\pi - \delta \qquad &\text{if} \quad \beta \neq 3, \quad \delta > 0 \end{aligned} \tag{3}$$

If only the dominant terms are of interest, we get the following results

$$
\begin{gathered}
{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\, z\right) \sim \frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)} K_{p,q}(z), \\
0 \leqslant p \leqslant q-1, \qquad |z| \to \infty, \qquad |\arg z| \leqslant \pi - \delta, \qquad \delta > 0. \qquad (4)
\end{gathered}
$$

$$
\begin{gathered}
{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|\, -z\right) \sim \frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)} \{K_{p,q}(ze^{i\pi}) + K_{p,q}(ze^{-i\pi})\}, \\
0 \leqslant p < q-1, \qquad \text{that is,} \qquad \beta \geqslant 3; \\
z \to +\infty, \qquad \text{that is,} \quad \arg z = 0. \qquad (5)
\end{gathered}
$$

The case $\beta = 2$ has many important applications. We record the asymptotic expansion for this situation in full:

$$
\begin{gathered}
{}_pF_{p+1}\left(\begin{matrix}\alpha_p\\ \rho_{p+1}\end{matrix}\middle|\, -z\right) \sim \frac{\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)} \{K_{p,p+1}(ze^{i\pi}) + K_{p,p+1}(ze^{-i\pi}) + L_{p,p+1}(z)\}, \\
|z| \to \infty, \qquad |\arg z| \leqslant 2\pi - \delta, \qquad \delta > 0. \qquad (6)
\end{gathered}
$$

To facilitate computations in (5), (6), see 5.11.1(21). For numerous applications it is convenient to replace z by $z^2/4$ and write

$$
\begin{gathered}
{}_pF_{p+1}\left(\begin{matrix}\alpha_p\\ \rho_{p+1}\end{matrix}\middle|\, \frac{-z^2}{4}\right) \sim \frac{\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)} \{K_{p,p+1}([\tfrac{1}{2}ze^{i\pi/2}]^2) + K_{p,p+1}([\tfrac{1}{2}ze^{-i\pi/2}]^2) \\
+ L_{p,p+1}(z^2/4)\}, \\
|z| \to \infty, \qquad |\arg z| \leqslant \pi - \delta, \qquad \delta > 0. \qquad (7)
\end{gathered}
$$

Also

$$
\begin{gathered}
{}_pF_{p+1}\left(\begin{matrix}\alpha_p\\ \rho_{p+1}\end{matrix}\middle|\, \frac{z^2}{4}\right) \sim \frac{\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)} \{K_{p,p+1}([\tfrac{1}{2}z]^2) + K_{p,p+1}([\tfrac{1}{2}ze^{i\epsilon\pi}]^2) \\
+ L_{p,p+1}([\tfrac{1}{2}ze^{i\epsilon\pi}]^2)\}, \\
|z| \to \infty, \quad \delta - (2+\epsilon)\pi/2 \leqslant \arg z \leqslant (2-\epsilon)\pi/2 - \delta, \quad \epsilon = \pm 1, \quad \delta > 0. \quad (8)
\end{gathered}
$$

The apparent discrepancy in (8) when $|\arg z| < \pi/2$ is a case of Stoke's phenomenon.

Let us write

$$
\begin{gathered}
K_{p,p+1}([\tfrac{1}{2}z]^2) = \frac{e^z z^{2\gamma}}{2^{2\gamma+1}\pi^{1/2}} \sum_{k=0}^{\infty} d_k z^{-k}, \qquad d_0 = 1, \\
\gamma = \tfrac{1}{2}\left\{\tfrac{1}{2} + \sum_{h=1}^{p} \alpha_h - \sum_{h=1}^{p+1} \rho_h\right\} = \tfrac{1}{2}\{\tfrac{1}{2} + B_1 - C_1\}. \qquad (9)
\end{gathered}
$$

We record below recursion formulas for the d_k's in the cases $p = 0, 1,$ and 2 In this situation observe that

$$d_k = 2^k N_k \tag{10}$$

and N_k is given by 5 11 5(19) for $k = 1, 2, 3$

CASE 1 $p = 0$ If $\rho_1 = \nu + 1$,

$$d_k = \frac{(\frac{1}{2} + \nu)_k (\frac{1}{2} - \nu)_k}{2^k k!} \tag{11}$$

See the comment following 5 11 3(5)

CASE 2 $p = 1$

$$\begin{aligned} 2(k+1)\, d_{k+1} = {} & [3k^2 + 2k(1 + C_1 - 3B_1) + 4N_1]\, d_k \\ & - (k - 2\gamma - 1)(k - 2\gamma + 1 - 2\rho_1)(k - 2\gamma + 1 - 2\rho_2)\, d_{k-1}, \end{aligned} \tag{12}$$

where B_1, C_1, and N_1 are defined by 5 11 1(19, 20) There $q = p + 1$ and $\beta = 2$

CASE 3 $p = 2$.

$$\begin{aligned} 2(k+1)\, d_{k+1} = {} & [5k^2 + 2k(3 + B_1 - 3C_1 - 10\gamma) + 4N_1]\, d_k \\ & - [4k^3 - 6k^2(C_1 + 4\gamma) + 2k(24\gamma^2 + 12\gamma C_1 + C_1 + 4C_2 - 1) \\ & - 32\gamma^3 - 24\gamma^2 C_1 - 4\gamma(C_1 + 4C_2 - 1) + 2C_1 - 4C_2 - 8C_3 - 1]\, d_{k-1} \\ & + (k - 2\gamma - 2)(k - 2\gamma - 2\rho_1)(k - 2\gamma - 2\rho_2)(k - 2\gamma - 2\rho_3)\, d_{k-2}, \end{aligned} \tag{13}$$

where the notation is the same as in (12), 5 11 1(20), and $C_3 = \rho_1\rho_2\rho_3$

Next we write the complete asymptotic expansion for the case $\beta = 3$ We have

$${}_pF_{p+2}\left(\begin{matrix}\alpha_p \\ \rho_{p+2}\end{matrix}\middle| -z\right) \sim \frac{\Gamma(\rho_{p+2})}{\Gamma(\alpha_p)}\{U(z) + V(z) + L_{p,p+2}(z)\} \qquad |z| \to \infty, \tag{14}$$

where

$$U(z) = K_{p,p+2}(ze^{i\pi}) + K_{p,p+2}(ze^{-i\pi}),$$

[see 5 11 1(20)] and either

$$V(z) = V_1(z) = \Gamma^{(1)}{}_{p+3}(1)\, K_{p,p+2}(ze^{-3i\pi}),$$

or

$$V(z) = V_2(z) = \Gamma^{(1)}{}_{p+3}(1)\, K_{p,p+2}(ze^{3i\pi}), \tag{15}$$

according as $\delta - 3\pi/2 \leqslant \arg z \leqslant 2\pi - \delta$ or $\delta - 2\pi \leqslant \arg z \leqslant 3\pi/2 - \delta$, respectively, $\delta > 0$. The apparent discrepancy in (15) when $|\arg z| < 3\pi/2$ is a case of Stoke's phenomenon. It is readily shown that

$$V_1(z) = \frac{[\sum_{j=1}^{p} \exp(-2i\pi\alpha_j) - \sum_{j=1}^{p+2} \exp(-2i\pi\rho_j)]}{(2\pi)\, 3^{1/2}} \exp\{-3z^{1/3}\}\, z^{\gamma} e^{-3i\pi\gamma}$$
$$\times \sum_{r=0}^{\infty} (-)^r N_r z^{-r/3}. \tag{16}$$

When $\beta = 4$,

$${}_pF_{p+3}\left(\begin{matrix}\alpha_p \\ \rho_{p+3}\end{matrix}\middle| -z\right) \sim \frac{\Gamma(\rho_{p+3})}{\Gamma(\alpha_p)} \{A(z) + B(z) + L_{p,p+3}(z)\},$$
$$|z| \to \infty, \qquad |\arg z| \leqslant 2\pi - \delta, \qquad \delta > 0, \tag{17}$$

where

$$A(z) = K_{p,p+3}(ze^{i\pi}) + K_{p,p+3}(ze^{-i\pi}),$$

[see 5.11.1(21)] and

$$B(z) = \Gamma^{1,p}_{p+4}(1)\, K_{p,p+3}(ze^{-3i\pi}) + \bar{\Gamma}^{1,p}_{p+4}(1)\, K_{p,p+3}(ze^{3i\pi})$$
$$= \frac{\exp\{-4z^{1/4}\cos\pi/4\}\, z^{\gamma}}{(2\pi)^{3/2}} \sum_{r=0}^{\infty} (-)^r N_r z^{-r/4} b_r(z),$$

$$b_r(z) = \left\{\sum_{j=1}^{p} \cos 2\pi\alpha_j - \sum_{j=1}^{p+3} \cos 2\pi\rho_j\right\} \cos((\pi r/4) + 3\pi\gamma + 4z^{1/4}\sin\pi/4)$$
$$- \left\{\sum_{j=1}^{p} \sin 2\pi\alpha_j - \sum_{j=1}^{p+3} \cos 2\pi\rho_j\right\} \sin((\pi r/4) + 3\pi\gamma + 4z^{1/4}\sin\pi/4). \tag{18}$$

5.11.3. Asymptotic Expansion of ${}_pF_p(z)$ for Large z

$${}_pF_p\left(\begin{matrix}\alpha_p \\ \rho_p\end{matrix}\middle| -z\right) \sim \frac{\Gamma(\rho_p)}{\Gamma(\alpha_p)} \{K_{p,p}(ze^{-i\pi}) + L_{p,p}(z)\},$$
$$|z| \to \infty, \qquad \delta - \pi/2 \leqslant \arg z \leqslant 3\pi/2 - \delta, \qquad \delta > 0. \tag{1}$$

$${}_pF_p\left(\begin{matrix}\alpha_p \\ \rho_p\end{matrix}\middle| -z\right) \sim \frac{\Gamma(\rho_p)}{\Gamma(\alpha_p)} \{K_{p,p}(ze^{i\pi}) + L_{p,p}(z)\},$$
$$|z| \to \infty, \qquad \delta - 3\pi/2 \leqslant \arg z \leqslant \pi/2 - \delta, \qquad \delta > 0. \tag{2}$$

We may also write

$${}_pF_p\left(\begin{matrix}\alpha_p \\ \rho_p\end{matrix}\middle| z\right) \sim \frac{\Gamma(\rho_p)}{\Gamma(\alpha_p)} \{K_{p,p}(z) + L_{p,p}(ze^{i\epsilon\pi})\}, \qquad |z| \to \infty,$$
$$\delta - (2+\epsilon)\pi/2 \leqslant \arg z \leqslant (2-\epsilon)\pi/2 - \delta, \qquad \epsilon = \pm 1, \qquad \delta > 0. \tag{3}$$

If $|\arg z| < \pi/2$, the apparent discrepancy in (1), (2) and also in (3) is again a case of Stoke's phenomenon

We now put

$$K_{p,p}(2z) = e^{2z}(2z)^{\gamma} \sum_{k=0}^{\infty} d_k z^{-k}, \quad d_0 = 1, \quad \gamma = \sum_{h=1}^{p} (\alpha_h - \rho_h) = B_1 - C_1. \tag{4}$$

In the sequel, we give the recursion formulas for d_k for the cases $p = 1$ and $p = 2$

CASE 1 $p = q = 1$

$$d_k = \frac{(\rho_1 - \alpha_1)_k (1 - \alpha_1)_k}{2^k k!} \tag{5}$$

Note that if $\rho_1 = 2\alpha_1 = 2\nu + 1$, then (5) and 5 11 2(11) are the same and the d_k's are coefficients associated with the asymptotic expansion of $I_\nu(z)$ [see 6 2 7(37)]

CASE 2 $p = q = 2$

$$\begin{aligned} 4(k+1)\, d_{k+1} = {} & 2[2k^2 - k(2\gamma + B_1 - 1) + N_1]\, d_k \\ & - (k - 1 - \gamma)(k - \rho_1 - \gamma)(k - \rho_2 - \gamma)\, d_{k-1}, \end{aligned} \tag{6}$$

where N_1 is given by 5 11 1(20), with $p = q = 2$ and $\beta = 1$ If $\alpha_2 = 1$, $\rho_1 = \alpha_1 + \nu + \frac{1}{2}$, $\rho_2 = \alpha_1 - \nu + \frac{1}{2}$, then d_k is given by 5 11 2(11)

5 11 4 Further Representations of ${}_pF_q(z)$ for Large z, $0 \leqslant p \leqslant q - 1$

We now give some results which follow from the analysis of ${}_{p+2}F_q(-n, n + \lambda, a_p, b_q, z)$ for large n given in 7 4 5 and 7.4 6

This is accomplished using the confluence principle Thus from 3 5(33–35) and 7 4 5(4),

$${}_pF_q\left(\begin{matrix}\alpha_p \\ \rho_q\end{matrix}\middle| -z\right) = \lim_{n\to\infty} {}_{p+2}F_q\left(\begin{matrix}-n, n + \lambda\ \alpha_p \\ \rho_q\end{matrix}\middle| \frac{z}{n(n+\lambda)}\right), \tag{1}$$

$$\begin{aligned} {}_pF_q\left(\begin{matrix}\alpha_p \\ \rho_q\end{matrix}\middle| -z\right) \sim {} & \frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)} L_{p,q}(z) + \frac{2(2\pi)^{(1-\beta)/2}\Gamma(\rho_q)\, z^{\gamma}}{\beta^{1/2}\Gamma(\alpha_p)} \\ & \times \exp\{\beta z^{1/\beta} \cos \pi/\beta + \lambda_3 z^{-1/\beta} \cos \pi/\beta + O(z^{-2/\beta})\} \\ & \times \cos\{\beta z^{1/\beta} \sin \pi/\beta + \pi\gamma - \lambda_3 z^{-1/\beta} \sin \pi/\beta + O(z^{-2/\beta})\} \\ & + (\beta - 2) \quad \text{exponentially lower order terms,} \end{aligned}$$

$$\beta = q + 1 - p \geqslant 3, \quad |z| \to \infty, \quad |\arg z| \leqslant \pi - \delta, \quad \delta > 0 \tag{2}$$

Here γ and λ_3 are defined in 7 4 5(5) Also, γ is given by 5 11 1(19) and λ_3 is simply N_1 of 5 11 1(20)

Similarly, from 7.4.5(6), we have

$$
\begin{aligned}
{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right) &= \lim_{n\to\infty} {}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p\\ \rho_q\end{matrix}\middle| -\frac{z}{n(n+\lambda)}\right) \sim \frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)} L_{p,q}(ze^{\epsilon\pi i}) \\
&\quad + \frac{(2\pi)^{(1-\beta)/2}\Gamma(\rho_q)\, z^{\gamma}}{\beta^{1/2}\Gamma(\alpha_p)} \exp\{\beta z^{1/\beta} + \lambda_3 z^{-1/\beta} + O(z^{-2/\beta})\} \\
&\quad + (\beta - 1) \quad \text{exponentially lower order terms,} \\
\beta = q + 1 - p \geqslant 3, &\qquad |z| \to \infty, \qquad |\arg z| \leqslant \pi - \delta, \qquad \delta > 0, \\
&\qquad \epsilon = +(-) \quad \text{if} \quad \arg z \leqslant (>)\, 0,
\end{aligned} \tag{3}
$$

and γ and λ_3 are as in (2).

From 7.4.6(2, 4), we get

$$
\begin{aligned}
{}_pF_{p+1}\left(\begin{matrix}\alpha_p\\ \rho_{p+1}\end{matrix}\middle| -z\right) &\sim \frac{\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)} L_{p,p+1}(z) + \frac{\Gamma(\rho_{p+1})\, z^{\gamma}}{\pi^{1/2}\Gamma(\alpha_p)} \exp\{\omega_2 z^{-1} + O(z^{-2})\} \\
&\quad \times \cos\{2z^{1/2} + \pi\gamma - \omega_1 z^{-1/2} - \omega_5 z^{-3/2} + O(z^{-5/2})\}, \\
&\quad |z| \to \infty, \qquad \arg z \leqslant \pi - \delta, \qquad \delta > 0,
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
{}_pF_{p+1}\left(\begin{matrix}\alpha_p\\ \rho_{p+1}\end{matrix}\middle| z\right) &\sim \frac{\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)} L_{p,p+1}(ze^{\epsilon\pi i}) + \frac{\Gamma(\rho_{p+1})\, z^{\gamma}}{\pi^{1/2}\Gamma(\alpha_p)} \exp\{-\omega_2 z^{-1} + O(z^{-2})\} \\
&\quad \times \cosh\{2z^{1/2} + \omega_1 z^{-1/2} - \omega_5 z^{-3/2} + O(z^{-5/2})\} \\
&\quad |z| \to \infty, \qquad |\arg z| \leqslant \pi - \delta, \qquad \delta > 0, \qquad \epsilon \quad \text{as in (3)},
\end{aligned} \tag{5}
$$

where γ, ω_1, ω_2, and ω_5 are defined in 7.4.7(3). Also, γ and $\omega_1 = N_1$ are given by 5.11.1(19, 20), respectively, with $\beta = 2$.

Since the case $p = 0$ of (5) is of particular interest in connection with Bessel functions, we develop that case further by recourse to the differential equation approach employed in 7.4.1. Thus assume that the differential equation satisfied by ${}_0F_1(\ ;1+\nu;-z^2/4)$ [see 5.1(1, 2)] has a solution of the form

$$
K \exp\left\{c_0 z + 2c_1 \log z - \sum_{m=2}^{\infty} \frac{c_m 2^m}{(m-1)} z^{1-m}\right\} \tag{6}
$$

where K and c_m are constants. Put (6) into this differential equation. Equate the coefficients of powers of z^{-1} to zero and get the recursion formulas

$$
\begin{aligned}
c_0^2 + 1 &= 0, \\
(2\nu + 1 - m)\, c_m + 2 \sum_{l=0}^{m+1} c_l c_{m+1-l} &= 0.
\end{aligned} \tag{7}
$$

The two roots of the characteristic equation yield two different generally divergent series solutions of the differential equation Comparison of a linear combination of the two possible solutions of the form (6) with the known asymptotic expansion of $J_\nu(z)$ for large z then yields the formula

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \exp\{A_\nu(z)\} \cos\left\{B_\nu(z) - \frac{\pi\nu}{2} - \frac{\pi}{4}\right\}$$
$$|z| \to \infty \qquad |\arg(z)| \leqslant \pi - \delta \qquad \delta > 0 \tag{8}$$

where

$$\begin{aligned} A_\nu(z) &= -\sum_{m=1}^{\infty} \frac{c_{2m+1}}{m}\left(\frac{2}{z}\right)^{2m} \\ &= \frac{(\mu-1)}{16z^2}\left\{1 + \frac{(\mu-13)}{8z^2} + \frac{(\mu^2-53\mu+412)}{48z^4} + O(z^{-6})\right\} \\ B(z) &= z\sum_{m=0}^{\infty} \frac{c_0 c_{2m}}{(2m-1)}\left(\frac{2}{z}\right)^{2m} \\ &= z + \frac{(\mu-1)}{8z}\left\{1 + \frac{(\mu-25)}{48z^2} + \frac{(\mu^2-114\mu+1073)}{640z^4}\right. \\ &\quad \left. + \frac{(5\mu^3-1535\mu^2+54703\mu-375733)}{128z^6} + O(z^{-8})\right\} \qquad \mu = 4\nu^2 \end{aligned} \tag{9}$$

Note that (8) is exact for $\nu = \pm\frac{1}{2}$ Also

$$\begin{aligned} A_\nu(-z) &= A_\nu(z) \qquad & A_{-\nu}(z) &= A_\nu(z) \\ B_\nu(-z) &= -B_\nu(z) \qquad & B_{-\nu}(z) &= B_\nu(z) \end{aligned} \tag{10}$$

Asymptotic expansions for other Bessel functions follow from 6 2 7(5 7 10) Thus

$$\begin{aligned} Y_\nu(z) &\sim \left(\frac{2}{\pi z}\right)^{1/2} \exp\{A(z)\} \sin\left\{B(z) - \frac{\pi\nu}{2} - \frac{\pi}{4}\right\} \\ H^{(1)}(z) &\sim \left(\frac{2}{\pi z}\right)^{1/2} \exp\left\{A_\nu(z) + i\left[B_\nu(z) - \frac{\pi\nu}{2} - \frac{\pi}{4}\right]\right\} \\ C_\nu(z) &\sim \left(\frac{2}{\pi z}\right)^{1/2} \exp\{A_\nu(z)\} \cos\left\{B(z) - \frac{\pi\nu}{2} - \frac{\pi}{4} + \alpha\right\} \\ &\qquad |\arg z| \leqslant \pi - \delta \qquad \delta > 0 \end{aligned} \tag{11}$$

$$C_\nu(z) = J(z)\cos\alpha - Y_\nu(z)\sin\alpha \tag{12}$$

It calls for remark that if z is held fixed in (8 11) one needs the additional restriction $|\nu| < |z|$ in order for the correction terms to remain small

The forms (8), (11) are useful to get zeros of the functions. Inverting the series $B_\nu(z) = w$, we have

$$z = \psi_\nu(w) = w - \frac{(\mu-1)}{8w}\left\{1 + \frac{(7\mu-31)}{48w^2} + \frac{(83\mu^2-982\mu+3779)}{1920w^4} + O(w^{-6})\right\}. \tag{13}$$

Thus, the jth positive zero of $C_\nu(z)$ occurs at $\psi_\nu[(\pi/4)(4j+2\nu-1)-\alpha]$, which corresponds to the McMahon expansions given in Watson (1945) and Olver (1960).

For a further application of the above results, we have

$$\begin{aligned} J_{-\nu}(\zeta) + J_\nu(\zeta) &\sim 2\cos(\pi\nu/2)(2/\pi\zeta)^{1/2}\exp\{A_\nu(\zeta)\}\cos\{B_\nu(\zeta)-(\pi/4)\}, \\ J_{-\nu}(\zeta) - J_\nu(\zeta) &\sim -2\sin(\pi\nu/2)(2/\pi\zeta)^{1/2}\exp\{A_\nu(\zeta)\}\sin\{B_\nu(\zeta)-(\pi/4)\}, \\ |\zeta| &\to \infty, \qquad |\arg\zeta| \leqslant \pi-\delta, \qquad \delta > 0. \end{aligned} \tag{14}$$

Thus the jth positive zeros of the first and second functions in (14) occur at $\psi_\nu[(\pi/4)(4j-1)]$ and $\psi_\nu[(\pi/4)(4j+1)]$, respectively. If $\nu = \frac{1}{3}$, (14) gives information on the Airy functions $A_i(-z)$ and $B_i(-z)$ [see 6.2.8(4, 5)].

5.11.5. Recursion Formulas for Coefficients in the Exponential Asymptotic Expansions for $G^{m,n}_{p,q}(z)$ and ${}_pF_q(z)$

Here we develop recursion formulas for the coefficients c_k, M_k, and N_k which enter into the expansions for ${}_pg_q(z)$, $H_{p,q}(z)$, and $K_{p,q}(z)$, respectively, defined by 2.11(28), 5.7(13), and 5.11.1(19), respectively.

It is sufficient to develop the recursion formula for c_k as the coefficients M_k and N_k are simply related to c_k. The relations are

$$\begin{aligned} \beta^{-k}c_k\left(p, q+1 \,\middle|\, \begin{matrix}\alpha_p \\ 1, \rho_q\end{matrix}\right) &= N_k = N_k\left(p, q \,\middle|\, \begin{matrix}\alpha_p \\ \rho_q\end{matrix}\right), \\ M_k = M_k\left(p, q+1 \,\middle|\, \begin{matrix}1-\alpha_p \\ 0, 1-\rho_q\end{matrix}\right) &= (-)^k N_k\left(p, q \,\middle|\, \begin{matrix}\alpha_p \\ \rho_q\end{matrix}\right). \end{aligned} \tag{1}$$

For the development of the recursion formula for c_k we follow E. M. Wright (1958). First, we notice from 2.11(28) that if

$$F(z) = \sum_{k=0}^{\infty} {}_pg_q(k)\, z^k, \qquad p \leqslant q, \quad |z| < 1 \quad \text{if} \quad p = q+1, \tag{2}$$

then under the same conditions

$$F(z) = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=0}^{q}\Gamma(\rho_j)}\, {}_{p+1}F_{q+1}\left(\begin{matrix}1, \alpha_p \\ \rho_0, \rho_q\end{matrix} \,\middle|\, z\right). \tag{3}$$

Let $K^*_{p,q}(z)$ stand for the formal relation

$$K^*_{p,q}(z) = \frac{(2\pi)^{(1-\beta)/2}}{\beta^{1/2}} [\exp\{\beta(z)^{1/\beta}\}](z)^{\gamma+(1-\rho_0)/\beta} \sum_{k=0}^{\infty} \frac{c_k z^{-k/\beta}}{\beta^k}, \tag{4}$$

β and γ as in 5 11 1(19) Then

$$F(z) \sim K^*_{p,q}(z) \quad p \leqslant q-1, \quad |z| \to \infty, \quad |\arg z| \leqslant \pi - \delta, \quad \delta > 0 \tag{5}$$

Observe that if $\rho_0 = 1$, (3) becomes a ${}_pF_q$ and (4) reduces to $K_{p,q}(z)$ Clearly $K^*_{p,p}(z)$ enters the asymptotic expansion of $F(z)$ when $p = q$ If $p \leqslant q$, $K^*_{p,q}(z)$ formally satisfies a nonhomogeneous differential equation of order $q+1$, see 5 1(12) So upon substituting (4) into this differential equation, we can discover a recursion formula to generate the c_k's This is the idea used by E M Wright

Let

$$T(t) = \prod_{j=0}^{q} (t + \omega_j), \qquad U(t) = \prod_{j=1}^{p} (t + \lambda_j),$$
$$\omega_j = \beta\rho_j - \beta + \beta\gamma + 1 - \rho_0, \qquad \lambda_j = \beta\alpha_j + \beta\gamma + 1 - \rho_0, \tag{6}$$

$$T_s(-k) = \sum_{r=0}^{s} \frac{(-)^{s-r} T(r-k)}{r!\,(s-r)!} = \frac{\Delta^s T(-k)}{s!}, \tag{7}$$

where Δ is the forward difference operator with respect to t and in (7), t is replaced by $-k$ after differencing We also need to define $U_s(-k)$, which is given by (7) with T replaced by U It may be shown that

$$\sum_{s=0}^{q+1} T_{q+1-s}(s-k)\, c_{k-s} - \sum_{s=0}^{p} U_{p-s}(s-k)\, c_{k-s} = 0, \tag{8}$$

where c_k is given by 2 11(29), that is,

$$c_k = c_k\left(p\ q+1 \,\middle|\, \begin{matrix} \alpha_p \\ \rho_0, \rho_q \end{matrix}\right) \tag{9}$$

In (8), $c_k = 0$ for $k < 0$ In view of the simple relations

$$T_{q+1}(-k) - U_p(-k) = 0 \qquad T_q(-k) - U_{p-1}(-k) = -\beta k \tag{10}$$

$$\beta k c_k = \sum_{s=1}^{q} T_{q-s}(s-k)\, c_{k-s} - \sum_{s=1}^{p-1} U_{p-s-1}(s-k)\, c_{k-s}, \tag{11}$$

where the second sum is nil if $p \leqslant 1$

If the largest k for which c_k is required is near to q, the coefficients in (11) can be readily evaluated by computing $T(t)$ for $t = q-1$, $q-2,\ldots, -k$ and then differencing these values up to $(q-1)$ times. If k is large with respect to q, then

$$T_{q-s}(s-k) = \sum_{r=0}^{s+1} \frac{(-)^{s+1-r}(q+1-r)!\,(k-r)!\,T_{q+1-r}(0)}{(k-s-1)!\,(q-s)!\,(s+1-r)!} \tag{12}$$

and so we need only to calculate $T_s(0)$ by differencing for $s = 0, 1,\ldots, q+1$. The formula analogous to (12) for $U_{p-s-1}(s-k)$ is obvious.

If k is small with respect to q, the above methods for the evaluation of the coefficients of c_r in (11) are not very efficient. In this event E. M. Wright proposes the following. From (6) and 2.9(13), we can write

$$U(t) = \sum_{r=0}^{p} S_r^{(p)}(\lambda)\, t^{p-r}, \qquad T(t) = \sum_{r=0}^{q+1} S_r^{(q+1)}(\omega)\, t^{q+1-r}. \tag{13}$$

In particular,

$$S_0^{(p)}(\lambda) = S_0^{(q+1)}(\omega) = 1,$$

$$S_1^{(p)}(\lambda) = \sum_{j=1}^{p} \lambda_j = \beta \sum_{j=1}^{p} \alpha_j + p\{\beta\gamma + 1 - \rho_0\}, \tag{14}$$

$$S_1^{(q+1)}(\omega) = \sum_{j=0}^{q} \omega_j = \beta \sum_{j=0}^{q} \rho_j + (q+1)\{-\beta + \beta\gamma + 1 - \rho_0\}.$$

Now

$$T(t-k) = \sum_{r=0}^{q+1} S_r^{(q+1)}(\omega) \sum_{v=0}^{q+1-r} (-)^v k^v \binom{q+1-r}{v} t^{q+1-r-v} \tag{15}$$

and so

$$T_{q+1-s}(-k) = \sum_{r=0}^{s} S_r^{(q+1)}(\omega) \sum_{v=0}^{s-r} (-)^v k^v \binom{q+1-r}{v}\binom{q+1-r-v}{q+1-s} B_{s-r-v}^{(s-q-1)}, \tag{16}$$

$$U_{p-s}(-k) = \sum_{r=0}^{s} S_r^{(p)}(\lambda) \sum_{v=0}^{s-r} (-)^v k^v \binom{p-r}{v}\binom{p-r-v}{p-s} B_{s-r-v}^{(s-p)}, \tag{17}$$

where $B_k^{(a)}(x)$ is defined in 2.8.

In certain situations, we can identify the coefficients N_k, for $k = 1, 2, 3$, with coefficients in 5.11.4(3, 4). For example, if $\exp(\lambda_3 z^{-1/\beta})$ is expanded

in the form $1 + \lambda_3 z^{-1/\beta} + O(z^{-2/\beta})$, and the exponential portion of 5 11 4(3) is compared with 5 11 2(4) and 5 11 1(19), we find that

$$N_1 = \lambda_3 \tag{18}$$

where λ_3 is given by 7 4 5(5) In a similar fashion, from 5 11 4(4) and the combination 5 11 2(6) and 5 11 1(21), we deduce that

$$N_1 = \omega_1 \quad N_2 = \omega_1^2/2 \quad \omega_2, \quad N_3 = \omega_1^3/6 - \omega_1\omega_2 \quad \omega_3, \quad q = p + 1, \tag{19}$$

where ω_1 and ω_2 are given in 7 4 6(3)

Riney (1956) and Van der Corput (1957, 1959) use essentially the functional equation of the gamma function to derive a recursive formula for the c_k's A more sophisticated solution to the same problem has been given by Riney (1958) and Nörlund (1960) They show that the general case $p \leqslant q$ can be deduced from the $p = q$ case, and that the particular case $p = q$ can be solved by considering the singular solution at $z = 1$ of the differential equation satisfied by

$${}_{q+2}F_{q+1}\left(\begin{matrix} 1 & \alpha_p, & \dfrac{\alpha+1}{\beta}, & \dfrac{\alpha+2}{\beta}, & , & \dfrac{\alpha+\beta}{\beta} \\ & & \rho_0, \rho_q & & & \end{matrix}\middle| z\right)$$

A recursion formula for the coefficients in the expansion of $[{}_pg_q(z)]^{-1}$ has also been studied by Van der Corput (1957, 1959) and Riney (1959)

Chapter VI IDENTIFICATION OF THE ${}_pF_q$ AND G-FUNCTIONS WITH THE SPECIAL FUNCTIONS OF MATHEMATICAL PHYSICS

6.1. Introduction

In Chapters III–V, we have developed numerous representations for the ${}_pF_q$ and its generalization, the G-function. A vast number of the special functions of mathematical physics are particular cases of this material. For instance, many properties of Bessel functions follow from Chapter IV. To facilitate use of our work in the applications, we identify the more important special functions with the ${}_pF_q$ and $G_{p,q}^{m,n}$ notation.

The exponential function e^z, the Bessel function $J_\nu(z)$, and the sine integral $\mathrm{Si}(z)$ are examples of named functions. Named functions are identified in terms of the ${}_pF_q$ notation in 6.2. Here we also delineate key properties of Bessel functions, Struve functions, and the important special cases of the incomplete gamma function which are useful for the applications and for our rational and Chebyshev approximations to these functions. It often happens that one has a ${}_pF_q$ and wishes to identify it as a named function. These data are set forth in 6.3.

In 6.4, named functions are expressed in terms of the G-function, while in 6.5 the G-function is expressed in terms of named functions.

6.2. Named Special Functions Expressed as ${}_pF_q$'s

6.2.1. Elementary Functions

$$e^z = {}_0F_0(z). \tag{1}$$

$$(1+z)^a = {}_1F_0(-a; -z) = {}_2F_1(-a, b; b; -z), \qquad |z| < 1. \tag{2}$$

$$(1+z)^{2a} + (1-z)^{2a} = 2\,{}_2F_1(-a, \tfrac{1}{2} - a; \tfrac{1}{2}; z^2), \qquad |z| < 1. \tag{3}$$

$$(1+z)^{2a} - (1-z)^{2a} = 4az\,{}_2F_1(\tfrac{1}{2} - a, 1 - a; \tfrac{3}{2}; z^2), \qquad |z| < 1. \tag{4}$$

$$[\tfrac{1}{2} + \tfrac{1}{2}(1 - z)^{1/2}]^{1-2a} = {}_2F_1(a, a - \tfrac{1}{2}, 2a, z), \qquad |z| < 1 \tag{5}$$

$$(1 + z)(1 - z)^{-2a-1} = {}_2F_1(a + 1, 2a, a, z), \qquad |z| < 1 \tag{6}$$

$$\sum_{k=0}^{m} \frac{(\alpha_p)_k z^k}{(\rho_q)_k k!} = \frac{(\alpha_p)_m z^m}{(\rho_q)_m m!}\, {}_{q+2}F_p \left(\begin{matrix} -m, 1 - m - \rho_q, 1 \\ 1 - m - \alpha_p \end{matrix} \middle| \frac{(-)^{p+q+1}}{z} \right) \tag{7}$$

$$\sum_{k=m+1}^{\infty} \frac{(\alpha_p)_k z^k}{(\rho_q)_k k!} = \frac{(\alpha_p)_{m+1} z^{m+1}}{(\rho_q)_{m+1}(m+1)!}\, {}_{p+1}F_{q+1} \left(\begin{matrix} \alpha_p + m + 1, 1 \\ \rho_q + m + 1, m + 2 \end{matrix} \middle| z \right),$$

$$p \leqslant q \quad \text{or} \quad p = q + 1 \quad \text{and} \quad |z| < 1 \tag{8}$$

$$\cos z = {}_0F_1(\ , \tfrac{1}{2}, -\tfrac{1}{4}z^2) \tag{9}$$

$$\sin z = z\, {}_0F_1(\ , \tfrac{3}{2}, -\tfrac{1}{4}z^2) \tag{10}$$

$$\cos 2az = {}_2F_1(-a, a, \tfrac{1}{2}, \sin^2 z) \qquad |\sin z| < 1, \tag{11}$$

$$\sin 2az = 2a \sin z\, {}_2F_1(\tfrac{1}{2} + a, \tfrac{1}{2} - a, \tfrac{3}{2}, \sin^2 z), \quad |\sin z| < 1 \tag{12}$$

$$z \csc z = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \sin^2 z), \qquad |\sin z| < 1 \tag{13}$$

$$\ln(1 + z) = z\, {}_2F_1(1, 1, 2, -z), \qquad |z| < 1 \tag{14}$$

$$\ln[(1 + z)/(1 - z)] = 2z\, {}_2F_1(\tfrac{1}{2}, 1, \tfrac{3}{2}, z^2), \qquad |z| < 1 \tag{15}$$

$$\arcsin z = z\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, z^2), \qquad |z| < 1 \tag{16}$$

$$\arctan z = z\, {}_2F_1(\tfrac{1}{2}, 1, \tfrac{3}{2}, -z^2), \qquad |z| < 1 \tag{17}$$

See also 9 5(5–7) Equations (3), (4), (5), (14), (16), (17) are also valid for $z = 1$, and (11), (12), (13) are valid for $z = \pi/2$ If $z = 1$ in (3), (4), require $R(a) > 0$

$$aB(a, b)\, x^{-a}(1 - x)^{-b} I_x(a, b) = {}_2F_1 \left(\begin{matrix} 1, a + b \\ a + 1 \end{matrix} \middle| x \right), \qquad |x| < 1,$$

$$B(a, b) = \int_0^1 t^{a-1}(1 - t)^{b-1}\, dt = \Gamma(a)\, \Gamma(b)/\Gamma(a + b),$$

$$B(a, b)\, I_x(a, b) = \int_0^x t^{a-1}(1 - t)^{b-1}\, dt \tag{18}$$

$I_x(a, b)$ is essentially the incomplete beta function

6 2 2 The Gaussian Hypergeometric Function

See Chapter III

6.2.3. Legendre Functions

$$P_\nu^\mu(z) = [\Gamma(1-\mu)]^{-1}[(z+1)/(z-1)]^{\mu/2} \times {}_2F_1(-\nu, \nu+1; 1-\mu; (1-z)/2), \tag{1}$$

$$Q_\nu^\mu(z) = \frac{e^{i\pi\mu}\pi^{1/2}\Gamma(\mu+\nu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} z^{-\mu-\nu-1}(z^2-1)^{\mu/2} \times {}_2F_1((\mu+\nu+1)/2, (\mu+\nu+2)/2; \nu+\tfrac{3}{2}; z^{-2}), \tag{2}$$

where the complex z-plane is cut along the real axis from -1 to 1.

$$P_\nu^\mu(x) = [\Gamma(1-\mu)]^{-1}[(1+x)/(1-x)]^{\mu/2} \times {}_2F_1(-\nu, \nu+1; 1-\mu; (1-x)/2), \qquad -1 < x < 1. \tag{3}$$

$$Q_\nu^\mu(x) = \tfrac{1}{2}e^{-i\pi\mu}[e^{-i\pi\mu/2}Q_\nu^\mu(x+i0) + e^{i\pi\mu/2}Q_\nu^\mu(x-i0)], \qquad -1 < x < 1. \tag{4}$$

For references on Legendre functions, see Abramowitz and Stegun (1964), Erdélyi *et al.* (1953, Vol. 1, Chapter 3), Hobson (1955), Robin (1957), and Snow (1952).

6.2.4. Orthogonal Polynomials

See Chapter VIII.

6.2.5. Complete Elliptic Integrals

$$\mathrm{K}(k) = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{-1/2}\,d\theta = \tfrac{1}{2}\pi\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2), \qquad |k^2| < 1. \tag{1}$$

$$\mathrm{E}(k) = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{1/2}\,d\theta = \tfrac{1}{2}\pi\,{}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2), \qquad |k^2| < 1. \tag{2}$$

For references on complete and incomplete elliptic integrals and related topics, see Abramowitz and Stegun (1964), Byrd and Friedman (1954), Cayley (1961), Erdélyi *et al.* (1953, Vol. 2), Milne-Thomson (1950), Neville (1951), Oberhettinger and Magnus (1949), and Tricomi (1948).

6.2.6. Confluent Hypergeometric Functions and Whittaker Functions

In addition to the material given here, see Chapter IV, and in particular 4.9.

$${}_1F_1\left(\begin{matrix}a\\c\end{matrix}\middle|\, z\right) {}_1F_1\left(\begin{matrix}a\\c\end{matrix}\middle|\, -z\right) = {}_2F_3\left(\begin{matrix}a, c-a\\c, \frac{1}{2}c, \frac{1}{2}(c+1)\end{matrix}\middle|\, \frac{z^2}{4}\right). \tag{1}$$

The so-called Coulomb wave functions are essentially confluent hypergeometric functions Thus,

$$F_L(\eta, \rho) = C_L(\eta)\, \rho^{L+1} e^{-i\rho}\, {}_1F_1\left(\begin{matrix} L+1-i\eta \\ 2L+2 \end{matrix} \middle| 2i\rho\right),$$
$$C_L(\eta) = \frac{2^L e^{-\pi\eta/2}\, |\Gamma(L+1+i\eta)|}{(2L+1)!} \tag{2}$$

In view of 4 4(12), $F_L(\eta, \rho)$ is real if ρ, η, and L are real In the applications L is usually a positive integer or zero

$$F_L(\eta\ \rho) + iG_L(\eta, \rho) = \frac{(-)^{L+1}\Gamma(L+1-i\eta)\, e^{\pi\eta/2}(2\rho)^{L+1}e^{-i\rho}}{|\Gamma(L+1-i\eta)|} \times \psi(L+1-i\eta, 2L+2, 2i\rho) \tag{3}$$

$$F_L(\eta\ \rho) + iG_L(\eta\ \rho) = \frac{\Gamma(L+1-i\eta)\, e^{\pi\eta/2}\, {}_1L+1}{|\Gamma(L+1-i\eta)|} W_{i\eta\ L+\frac{1}{2}}(2i\rho) \tag{4}$$

For further material on Coulomb wave functions, see Abramowitz and Stegun (1964) and Curtis (1964) We have followed the notation of the former

The parabolic cylinder function is defined by

$$\begin{aligned} D_\nu(z) &= 2^{(\nu-1)/2} e^{-z^2/4} z\psi((1-\nu)/2, 3/2, z^2/2) \\ &= 2^{(\nu+1/2)/2} z^{-1/2} W_{(\nu+1/2)/2\ -1/4}(z^2/2) \\ &= 2^{\nu/2} e^{-z^2/4} \left[\frac{\Gamma(1/2)}{\Gamma[(1-\nu)/2]}\, {}_1F_1\left(\begin{matrix} -\nu/2 \\ 1/2 \end{matrix} \middle| z^2/2\right) \right. \\ &\quad \left. + \frac{z\Gamma(-1/2)}{2^{1/2}\Gamma(-\nu/2)}\, {}_1F_1\left(\begin{matrix} (1-\nu)/2 \\ 3/2 \end{matrix} \middle| z^2/2\right)\right] \end{aligned} \tag{5}$$

For further material on (5), see Erdelyi *et al* (1953, Vol 2, Chapter 8) J C P Miller (1955), and Abramowitz and Stegun (1964)

6 2 7 Bessel Functions

Definitions, Connecting Relations, and Power Series

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\, {}_0F_1(\ , 1+\nu, -z^2/4) \tag{1}$$

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\, {}_0F_1(\ , 1+\nu\ z^2/4) = e^{-i\nu\pi/2} J_\nu(ze^{i\pi/2}), \quad -\pi < \arg z \leqslant \pi/2 \tag{2}$$

$$J_\nu(z) = \frac{(z/2)^\nu e^{\mp iz}}{\Gamma(\nu+1)}\, {}_1F_1(\tfrac{1}{2}+\nu;\, 1+2\nu;\, \pm 2iz). \tag{3}$$

$$\begin{aligned} I_\nu(z) &= \frac{(z/2)^\nu e^{\mp z}}{\Gamma(\nu+1)}\, {}_1F_1(\tfrac{1}{2}+\nu;\, 1+2\nu;\, \pm 2z) \\ &= [\Gamma(\nu+1)\, 2^{2\nu}(2z)^{1/2}]^{-1}\, M_{0,\nu}(2z). \end{aligned} \tag{4}$$

$$Y_\nu(z) = (\csc \nu\pi)[(\cos \nu\pi)\, J_\nu(z) - J_{-\nu}(z)]. \tag{5}$$

$$\pi Y_n(z) = \left[\frac{\partial J_\nu(z)}{\partial \nu} - (-)^n \frac{\partial J_{-\nu}(z)}{\partial \nu}\right]_{\nu=n}, \qquad (\pi/2)\, Y_0(z) = \left[\frac{\partial J_\nu(z)}{\partial \nu}\right]_{\nu=0}. \tag{6}$$

$$C_\nu(z) = A J_\nu(z) + B Y_\nu(z), \tag{7}$$

where A and B are independent of z.

$$\begin{aligned} K_\nu(z) &= (\pi/2)(\csc \nu\pi)[I_{-\nu}(z) - I_\nu(z)] \\ &= \pi^{1/2} e^{-z}(2z)^\nu\, \psi(\tfrac{1}{2}+\nu;\, 1+2\nu;\, 2z) = (\pi/2z)^{1/2}\, W_{0,\nu}(2z). \end{aligned} \tag{8}$$

$$2K_n(z) = (-)^n \left[\frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=n}, \qquad K_0(z) = -\left[\frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=0}. \tag{9}$$

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \qquad H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z). \tag{10}$$

$$\begin{aligned} K_\nu(z) &= \tfrac{1}{2}\pi i \epsilon e^{i\epsilon\nu\pi/2} H_\nu^{(3-\epsilon)/2}(z e^{i\epsilon\pi/2}), \\ &\quad -(3+\epsilon)\,\pi/4 < \arg z \leqslant (3-\epsilon)\,\pi/4, \quad \epsilon = \pm 1. \end{aligned} \tag{11}$$

$$Y_\nu(z e^{im\pi}) = e^{-im\nu\pi} Y_\nu(z) + 2i(\sin m\nu\pi)(\cot \nu\pi)\, J_\nu(z), \tag{12}$$

$$K_\nu(z e^{im\pi}) = e^{-im\nu\pi} K_\nu(z) - \frac{i\pi(\sin m\nu\pi)}{\sin \nu\pi}\, I_\nu(z), \tag{13}$$

where m is an integer or zero.

$$\begin{aligned} Y_n(z) &= (2/\pi)[\gamma + \ln(z/2)]\, J_n(z) \\ &\quad - \frac{(z/2)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z^2}{4}\right)^k \\ &\quad - \frac{(z/2)^n}{\pi} \sum_{k=0}^{\infty} (-)^k \frac{[\psi(k+1) + \psi(n+k+1) - 2\psi(1)]}{k!\,(n+k)!} \left(\frac{z^2}{4}\right)^k. \end{aligned} \tag{14}$$

$$\begin{aligned} K_n(z) &= (-)^{n+1}[\gamma + \ln(z/2)]\, I_n(z) \\ &\quad + (1/2)(z/2)^{-n} \sum_{k=0}^{n-1} \frac{(-)^k (n-k-1)!}{k!} \left(\frac{z^2}{4}\right)^k \\ &\quad + \frac{(-)^n}{2} (z/2)^n \sum_{k=0}^{\infty} \frac{[\psi(k+1) + \psi(n+k+1) - 2\psi(1)]}{k!\,(n+k)!} \left(\frac{z^2}{4}\right)^k. \end{aligned} \tag{15}$$

Difference–Differential Formulas

Let $W_\nu(z)$ represent any of the Bessel functions of the first three kinds or the modified Bessel functions With each $W_\nu(z)$ we associate two parameters a and b as outlined in the following tabulation

$W_\nu(z)$	a	b
$J_\nu(z)$ $Y_\nu(z)$ $H_\nu^{(1)}(z)$ $H_\nu^{(2)}(z)$	1	1
$I_\nu(z)$	1	1
$K_\nu(z)$	1	1

(16)

$$aW_{\nu+1}(z) + bW_{\nu-1}(z) = (2\nu/z)\, W_\nu(z) \tag{17}$$

$$-aW_{\nu+1}(z) + bW_{\nu-1}(z) = 2W_\nu'(z) \tag{18}$$

$$zW_\nu'(z) + \nu W_\nu(z) = bzW_{\nu-1}(z) \tag{19}$$

$$zW_\nu'(z) - \nu W_\nu(z) = -azW_{\nu+1}(z) \tag{20}$$

$$(z^{-1}d/dz)^m\{z^\nu W_\nu(z)\} = b^m z^{\nu-m} W_{\nu-m}(z) \tag{21}$$

$$(z^{-1}d/dz)^m\{z^{-\nu} W_\nu(z)\} = (-a)^m z^{-\nu-m} W_{\nu+m}(z) \tag{22}$$

$$W_\nu^{(m)}(z) = 2^{-m} \sum_{k=0}^{m} (\ )^k \binom{m}{k} a^k b^{m-k} W_{\nu-m+2k}(z) \tag{23}$$

$$[z^2D^2 + zD + (abz^2 - \nu^2)]\, W_\nu(z) = 0 \qquad D = d/dz \tag{24}$$

If

$$y(z) = z^{-\alpha} e^{-f(z)}\, C_\nu[h(z)]$$

then with $y' = y'(z)$ $h = h(z)$ and $f = f(z)$,

$$\begin{aligned} z^2y'' + z\{2\alpha + z[2f' - (h''/h') + (h'/h)]\}\, y' + \{[\alpha(\alpha-1) + 2\alpha f'z + f''z^2 \\ + (f')^2z^2 - (h''/h')(\alpha z + f'z^2)] \\ + (h'/h)(\alpha z + f'z^2) \\ + (h')^2(z^2 - \nu^2z^2/h^2)\}\, y = 0 \end{aligned} \tag{25}$$

Wronskians

We define

$$W\{u(z),\ v(z)\} = u(z)\, v'(z) - u'(z)\, v(z) \tag{26}$$

Then

$$W\{J_\nu(z), J_{-\nu}(z)\} = -(2/\pi z)\sin\nu\pi, \tag{27}$$

$$W\{J_\nu(z), Y_\nu(z)\} = 2/\pi z, \tag{28}$$

$$W\{H_\nu^{(1)}(z), H_\nu^{(2)}(z)\} = -4i/\pi z, \tag{29}$$

$$W\{I_\nu(z), I_{-\nu}(z)\} = -(2/\pi z)\sin\nu\pi, \tag{30}$$

$$W\{I_\nu(z), K_\nu(z)\} = -1/z. \tag{31}$$

Asymptotic Expansions

$$H_\nu^{(1)}(z) \sim (2/\pi z)^{1/2}\exp[i(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)]\,{}_2F_0\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu; \frac{1}{2iz}\right),$$
$$|z|\to\infty, \qquad \delta - \pi \leqslant \arg z \leqslant 2\pi - \delta, \qquad \delta > 0. \tag{32}$$

$$H_\nu^{(2)}(z) \sim (2/\pi z)^{1/2}\exp[-i(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)]\,{}_2F_0\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu; -\frac{1}{2iz}\right),$$
$$|z|\to\infty, \qquad \delta - 2\pi \leqslant \arg z \leqslant \pi - \delta, \qquad \delta > 0. \tag{33}$$

$$J_\nu(z) = (2/\pi z)^{1/2}[A(z)\cos\theta + B(z)\sin\theta], \tag{34}$$

$$Y_\nu(z) = (2/\pi z)^{1/2}[A(z)\sin\theta - B(z)\cos\theta],$$
$$\theta = z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi, \tag{35}$$

$$A(z) \sim {}_4F_1\left(\begin{matrix}\tfrac{1}{4}+\tfrac{1}{2}\nu, \tfrac{1}{4}-\tfrac{1}{2}\nu, \tfrac{3}{4}+\tfrac{1}{2}\nu, \tfrac{3}{4}-\tfrac{1}{2}\nu \\ \tfrac{1}{2}\end{matrix}\,\middle|\, -\frac{1}{z^2}\right),$$

$$B(z) \sim \frac{(\frac{1}{4}-\nu^2)}{2z}\,{}_4F_1\left(\begin{matrix}\tfrac{3}{4}+\tfrac{1}{2}\nu, \tfrac{3}{4}-\tfrac{1}{2}\nu, \tfrac{5}{4}+\tfrac{1}{2}\nu, \tfrac{5}{4}-\tfrac{1}{2}\nu \\ \tfrac{3}{2}\end{matrix}\,\middle|\, -\frac{1}{z^2}\right),$$
$$|z|\to\infty, \qquad |\arg z| \leqslant \pi - \delta, \qquad \delta > 0. \tag{36}$$

$$I_\nu(z) \sim \frac{e^z}{(2\pi z)^{1/2}}\,{}_2F_0\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu; \frac{1}{2z}\right)$$
$$+ \frac{\exp[-z - \epsilon(\frac{1}{2}+\nu)\,i\pi]}{(2\pi z)^{1/2}}\,{}_2F_0\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu; -\frac{1}{2z}\right),$$
$$|z|\to\infty,\ \delta - (2+\epsilon)\pi \leqslant \arg z \leqslant (2-\epsilon)\pi - \delta,$$
$$\delta > 0,\ \epsilon = \pm 1. \tag{37}$$

$$K_\nu(z) \sim (\pi/2z)^{1/2}\,e^{-z}\,{}_2F_0\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu; -\frac{1}{2z}\right),$$
$$|z|\to\infty, \quad |\arg z| \leqslant 3\pi/2 - \delta, \quad \delta > 0. \tag{38}$$

See 5.11.4(6–14) for some other asymptotic expansions.

If ν is half an odd integer, the hypergeometric series in (32)–(38) terminate so that we may replace asymptotic equality by equality and omit the restrictions on arg z

Products of Bessel Functions

$$J_\mu(z)\,J_\nu(z) = \frac{(z/2)^{\mu+\nu}}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\,{}_2F_3\left(\begin{matrix}(\mu+\nu+1)/2,\,(\mu+\nu+2)/2\\ \mu+1,\,\nu+1,\,\mu+\nu+1\end{matrix}\,\middle|\,-z^2\right), \tag{39}$$

$$J_\nu(z)\,J_{\nu+1}(z) = \frac{(z/2)^{2\nu+1}}{\Gamma(\nu+1)\,\Gamma(\nu+2)}\,{}_1F_2\left(\begin{matrix}\nu+3/2\\ \nu+2,\,2\nu+2\end{matrix}\,\middle|\,-z^2\right), \tag{40}$$

$$J_\nu^2(z) = \frac{(z/2)^{2\nu}}{[\Gamma(\nu+1)]^2}\,{}_1F_2(\nu+\tfrac{1}{2},\,\nu+1,\,2\nu+1,\,-z^2) \tag{41}$$

$$J_{-\nu}(z)\,J_\nu(z) = \frac{(\sin\nu\pi)}{\nu\pi}\,{}_1F_2(\tfrac{1}{2},\,1+\nu,\,1-\nu,\,-z^2) \tag{42}$$

$$J_\nu(z)\,I_\nu(z) = \frac{(z/2)^{2\nu}}{[\Gamma(\nu+1)]^2}\,{}_0F_3\left(,\,\frac{\nu+1}{2},\,\frac{\nu+2}{2},\,\nu+1,\,\frac{-z^4}{64}\right) \tag{43}$$

$$\begin{aligned} J_{-\nu}(z)\,I_\nu(z) = {}&\frac{(\sin\nu\pi)}{\nu\pi}\,{}_0F_3\left(,\,1+\tfrac{1}{2}\nu,\,1-\tfrac{1}{2}\nu,\,\tfrac{1}{2},\,\frac{-z^4}{64}\right)\\ &-\frac{2(\sin\nu\pi)(z/2)^2}{\nu\pi(1-\nu^2)}\,{}_0F_3\left(,\,\frac{3+\nu}{2},\,\frac{3-\nu}{2},\,\frac{3}{2},\,\frac{-z^4}{64}\right) \end{aligned} \tag{44}$$

Miscellaneous

$$\left.\frac{\partial I_\nu(z)}{\partial\nu}\right|_{\nu=\pm1/2} = (2\pi z)^{-1/2}[e^z\,\mathrm{Ei}(-2z) \mp e^{-z}\,\mathrm{Ei}(2z)] \tag{45}$$

$$\begin{aligned} \left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=1/2} &= (2/\pi z)^{1/2}[\mathrm{Ci}(2z)\sin z - \mathrm{Si}(2z)\cos z],\\ \left.\frac{\partial J_\nu(z)}{\partial\nu}\right|_{\nu=-1/2} &= (2/\pi z)^{1/2}[\mathrm{Ci}(2z)\cos z + \mathrm{Si}(2z)\sin z] \end{aligned} \tag{46}$$

$$\mathrm{ber}_\nu(z) + i\,\mathrm{bei}_\nu(z) = J_\nu(ze^{3i\pi/4}) = e^{i\nu\pi}J_\nu(ze^{-i\pi/4}) \tag{47}$$

$$\mathrm{ker}_\nu(z) + i\,\mathrm{kei}_\nu(z) = \exp(-\tfrac{1}{2}i\pi\nu)\,K_\nu(ze^{i\pi/4}) = \tfrac{1}{2}i\pi H_\nu^{(1)}(ze^{3i\pi/4}) \tag{48}$$

On the left-hand side of (47), (48), the subscript is omitted when $\nu = 0$

Bessel functions of order 1/3 are essentially Airy functions, see 6 2 8

For further material on Bessel functions, see Watson (1945), Erdelyi *et al* (1953, Vol 2, Chapter 7), and Abramowitz and Stegun (1964)

6.2.8. Airy Functions

Let

$$\xi = (\tfrac{2}{3})\, z^{3/2}. \tag{1}$$

$$\mathrm{Ai}(z) = (z^{1/2}/3)\{I_{-1/3}(\xi) - I_{1/3}(\xi)\} = \pi^{-1}(z/3)^{1/2}\, K_{1/3}(\xi). \tag{2}$$

$$\mathrm{Bi}(z) = (z/3)^{1/2}\{I_{-1/3}(\xi) + I_{1/3}(\xi)\}. \tag{3}$$

$$\mathrm{Ai}(-z) = (z^{1/2}/3)\{J_{-1/3}(\xi) + J_{1/3}(\xi)\}. \tag{4}$$

$$\mathrm{Bi}(-z) = (z/3)^{1/2}\{J_{-1/3}(\xi) - J_{1/3}(\xi)\}. \tag{5}$$

$$\mathrm{Ai}'(z) = -(z/3)\{I_{-2/3}(\xi) - I_{2/3}(\xi)\} = -(z/3^{1/2}\pi)\, K_{2/3}(\xi). \tag{6}$$

$$\mathrm{Bi}'(z) = 3^{-1/2} z\{I_{-2/3}(\xi) + I_{2/3}(\xi)\}. \tag{7}$$

$$\mathrm{Ai}'(-z) = -(z/3)\{J_{-2/3}(\xi) - J_{2/3}(\xi)\}. \tag{8}$$

$$\mathrm{Bi}'(-z) = 3^{-1/2} z\{J_{-2/3}(\xi) + J_{2/3}(\xi)\}. \tag{9}$$

For a complete development of Airy functions and associated Airy functions, see Luke (1962a).

6.2.9. Lommel Functions, Struve Functions, and Associated Bessel Functions

Definitions, Connecting Relations, and Power Series

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} \times {}_1F_2\left(\begin{matrix}1\\ \tfrac{1}{2}(\mu-\nu+3),\ \tfrac{1}{2}(\mu+\nu+3)\end{matrix}\,\middle|\, \frac{-z^2}{4}\right). \tag{1}$$

$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + \{2^{\mu-1}\Gamma[\tfrac{1}{2}(\mu-\nu+1)]\,\Gamma[\tfrac{1}{2}(\mu+\nu+1)]\} \times \{\sin[(\mu-\nu)\pi/2]\, J_\nu(z) - \cos[(\mu-\nu)\pi/2]\, Y_\nu(z)\}. \tag{2}$$

$$\mathbf{H}_\nu(z) = \frac{(z/2)^{\nu+1}}{\Gamma(3/2)\,\Gamma(3/2+\nu)}\, {}_1F_2\left(\begin{matrix}1\\ 3/2,\ 3/2+\nu\end{matrix}\,\middle|\, \frac{-z^2}{4}\right) = [\pi 2^{\nu-1}(\tfrac{1}{2})_\nu]^{-1}\, s_{\nu,\nu}(z). \tag{3}$$

$$\mathbf{H}_\nu(z) - Y_\nu(z) = [\pi 2^{\nu-1}(\tfrac{1}{2})_\nu]^{-1}\, S_{\nu,\nu}(z). \tag{4}$$

$$\mathbf{L}_\nu(z) = \frac{(z/2)^{\nu+1}}{\Gamma(3/2)\,\Gamma(3/2+\nu)}\, {}_1F_2\left(\begin{matrix}1\\ 3/2,\ 3/2+\nu\end{matrix}\,\middle|\, \frac{z^2}{4}\right) = \exp[-\tfrac{1}{2}(\nu+1)\, i\pi]\, \mathbf{H}_\nu(ze^{i\pi/2}). \tag{5}$$

Equations (1) and (2) are Lommel functions $\mathbf{H}_\nu(z)$ is the Struve function and $\mathbf{L}_\nu(z)$ is the modified Struve function

$$\mathbf{J}_\nu(z) = \frac{\sin \nu\pi}{\pi} s_{0,\nu}(z) - \frac{\nu \sin \nu\pi}{\pi} s_{-1,\nu}(z) \tag{6}$$

$$\mathbf{E}_\nu(z) = -\frac{(1 + \cos \nu\pi)}{\pi} s_{0,\nu}(z) - \frac{\nu(1 - \cos \nu\pi)}{\pi} s_{-1,\nu}(z) \tag{7}$$

$$\pi[\mathbf{E}_n(z) + \mathbf{H}_n(z)] = \sum_{k=0}^{<\frac{1}{2}n} \frac{\Gamma(k + \frac{1}{2})(\frac{1}{2}z)^{n-2k-1}}{\Gamma(n + \frac{1}{2} - k)} \tag{8}$$

$$\pi[\mathbf{E}_{-n}(z) + \mathbf{H}_{-n}(z)] = (-)^{n+1} \sum_{k=0}^{<\frac{1}{2}n} \frac{\Gamma(n - k - \frac{1}{2})(\frac{1}{2}z)^{-n+2k+1}}{\Gamma(k + \frac{3}{2})} \tag{9}$$

$\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$ are known as Anger–Weber functions

Difference–Differential Properties

Both $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$ satisfy the following difference–differential equations

$$S_{\mu,\nu}(z) = S_{\mu,-\nu}(z) \tag{10}$$

$$S_{\mu+2,\nu}(z) = z^{\mu+1} - [(\mu + 1)^2 - \nu^2]\, S_{\mu,\nu}(z) \tag{11}$$

$$S'_{\mu,\nu}(z) + (\nu/z)\, S_{\mu,\nu}(z) = (\mu + \nu - 1)\, S_{\mu-1,\nu-1}(z) \tag{12}$$

$$S'_{\mu,\nu}(z) - (\nu/z)\, S_{\mu,\nu}(z) = (\mu - \nu - 1)\, S_{\mu-1,\nu+1}(z) \tag{13}$$

$$\begin{aligned}(2\nu/z)\, S_{\mu,\nu}(z) &= (\mu + \nu - 1)\, S_{\mu-1,\nu-1}(z) \\ &\quad - (\mu - \nu - 1)\, S_{\mu-1,\nu+1}(z)\end{aligned} \tag{14}$$

$$\begin{aligned}2S_{\mu,\nu}(z) &= (\mu + \nu - 1)\, S_{\mu-1,\nu-1}(z) \\ &\quad + (\mu - \nu - 1)\, S_{\mu-1,\nu+1}(z)\end{aligned} \tag{15}$$

$$[z^2D^2 + zD + (z^2 - \nu^2)]\, S_{\mu,\nu}(z) = z^{\mu+1}, \qquad D = d/dz \tag{16}$$

$$\mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = (2\nu/z)\, \mathbf{H}_\nu(z) + \frac{(z/2)^\nu}{\Gamma(1/2)\, \Gamma(\nu + 3/2)} \tag{17}$$

$$\mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\mathbf{H}_\nu(z) - \frac{(z/2)^\nu}{\Gamma(1/2)\, \Gamma(\nu + 3/2)} \tag{18}$$

$$z\mathbf{H}_\nu(z) + \nu\mathbf{H}_\nu(z) = z\mathbf{H}_{\nu-1}(z) \tag{19}$$

$$z\mathbf{H}_\nu(z) - \nu\mathbf{H}_\nu(z) = \frac{(z/2)^{\nu+1}}{\Gamma(3/2)\, \Gamma(\nu + 3/2)} - z\mathbf{H}_{\nu+1}(z) \tag{20}$$

$$[z^2D^2 + zD + z^2 - \nu^2]\, \mathbf{H}_\nu(z) = \frac{4(z/2)^{\nu+1}}{\Gamma(1/2)\, \Gamma(1/2 + \nu)} \tag{21}$$

Asymptotic Expansions

$$S_{\mu,\nu}(z) \sim z^{\mu-1}\,{}_3F_0(1, (1-\mu+\nu)/2, (1-\mu-\nu)/2; -4/z^2),$$
$$|z| \to \infty, \qquad |\arg z| \leqslant \pi - \delta, \qquad \delta > 0. \tag{22}$$

$$\mathrm{H}_\nu(z) - Y_\nu(z) \sim \frac{(z/2)^{\nu-1}}{\pi(1/2)_\nu}\,{}_3F_0(1, 1/2, 1/2-\nu; -4/z^2),$$
$$|z| \to \infty, \qquad |\arg z| \leqslant \pi - \delta, \qquad \delta > 0. \tag{23}$$

$$I_{-\nu}(z) - \mathrm{L}_\nu(z) \sim \frac{(z/2)^{\nu-1}}{\pi(1/2)_\nu}\,{}_3F_0(1, 1/2, 1/2-\nu; 4/z^2),$$
$$|z| \to \infty, \qquad |\arg z| \leqslant \pi/2 - \delta, \qquad \delta > 0. \tag{24}$$

The functions $S_{\mu,\nu}(z)$ and those related to it are often called associated Bessel functions. Two other associated Bessel functions are

$$h_{\mu,\nu}(z) = \frac{e^{-z}z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_2F_2\left(\begin{matrix}1, \mu+\frac{3}{2}\\ \mu-\nu+2, \mu+\nu+2\end{matrix}\middle|\, 2z\right), \tag{25}$$

$$H_{\mu,\nu}(z) = h_{\mu,\nu}(z) - \frac{\Gamma(\mu-\nu+1)\,\Gamma(\mu+\nu+1)}{2^\mu(\frac{3}{2})_\mu}$$
$$\times \left[I_\nu(z) + \frac{K_\nu(z)\sin(\nu-\mu)\pi}{\pi\cos\mu\pi}\right], \tag{26}$$

which satisfy the differential equation

$$[z^2D^2 + zD - (z^2+\nu^2)]\,H_{\mu,\nu}(z) = e^{-z}z^{\mu+1}. \tag{27}$$

$$H_{\mu,\nu}(z) \sim -\frac{e^{-z}z^\mu}{2\mu+1}\,{}_3F_1\left(\begin{matrix}1, -\mu+\nu, -\mu-\nu\\ \frac{1}{2}-\mu\end{matrix}\middle|\, \frac{-1}{2z}\right),$$
$$|z| \to \infty, \qquad |\arg z| \leqslant 3\pi/2 - \delta, \qquad \delta > 0. \tag{28}$$

Further material on Lommel and Struve functions is given in the references at the end of 6.2.7. Also, for the functions just named, and for a complete treatment of the functions (25), (26), see Luke (1962a).

6.2.10. Integrals of Bessel Functions and Struve Functions

$$\int_0^z t^\mu J_\nu(t)\,dt = \frac{z^{\mu+\nu+1}}{2^\nu(\mu+\nu+1)\,\Gamma(\nu+1)}\,{}_1F_2\left(\begin{matrix}\frac{1}{2}(\mu+\nu+1)\\ \frac{1}{2}(\mu+\nu+3), \nu+1\end{matrix}\middle|\, \frac{-z^2}{4}\right),$$
$$R(\mu+\nu) > -1. \tag{1}$$

$$\int_0^z t^\mu H_\nu^{(1)}(t)\,dt = (2^\mu/\pi)\,e^{\frac{1}{2}i(\mu-\nu)\pi}\,\Gamma[\tfrac{1}{2}(\mu-\nu+1)]\,\Gamma[\tfrac{1}{2}(\mu+\nu+1)]$$
$$+ (\mu+\nu-1)\,zH_\nu^{(1)}(z)\,S_{\mu-1,\nu-1}(z) - zH_{\nu-1}^{(1)}(z)\,S_{\mu,\nu}(z),$$
$$R(\mu\pm\nu) > -1. \tag{2}$$

See 6 2 9(2) for $S_{\mu,\nu}(z)$

$$\int_0^z e^{it}t^\mu J_\nu(t)\,dt = \frac{z^{\mu+1}(z/2)^\nu}{(\mu+\nu+1)\,\Gamma(\nu+1)}\,{}_2F_2\left(\begin{matrix}\frac{1}{2}+\nu,\ \mu+\nu+1\\ 2\nu+1,\ \mu+\nu+2\end{matrix}\middle|\,2iz\right),$$
$$R(\mu+\nu) > -1 \qquad (3)$$

$$\int_0^z e^{it}t^\mu H_\nu^{(1)}(t)\,dt = \frac{2}{\pi}\,\frac{e^{\frac{1}{2}i(\mu-\nu)\pi}\Gamma(\mu+\nu+1)\,\Gamma(\mu-\nu+1)}{2^\mu(\frac{3}{2})_\mu}$$
$$+ ze^{\frac{1}{2}i\mu\pi}[(\mu-\nu)(\mu+\nu+1)^{-1}\,H_\nu^{(1)}(z)\,H_{\mu,\nu+1}(ze^{-\frac{1}{2}i\pi})$$
$$+ iH_{\nu+1}^{(1)}(z)\,H_{\mu,\nu}(ze^{-\frac{1}{2}i\pi})] + (\mu+\nu+1)^{-1}\,e^{iz}z^{\mu+1}H_\nu^{(1)}(z),$$
$$R(\mu\pm\nu) > -1 \qquad (4)$$

See 6 2 9(26) for $H_{\mu,\nu}(z)$

$$\int_0^z t^\mu \mathbf{H}_\nu(t)\,dt = \frac{z^{\mu+\nu+2}}{2^{\nu-1}\pi(\mu+\nu+2)(\frac{3}{2})_\nu}$$
$$\times\,{}_2F_3\left(\begin{matrix}1,\ 1+(\mu+\nu)/2\\ 3/2,\ \nu+3/2,\ 2+(\mu+\nu)/2\end{matrix}\middle|\,\frac{-z^2}{4}\right),$$
$$R(\mu+\nu) > -2 \qquad (5)$$

With

$$j_{\alpha,\nu}(z) = [\Gamma(\alpha)]^{-1}\int_0^z (z-t)^{\alpha-1}\,J_\nu(t)\,dt, \qquad R(\alpha)>0, \qquad R(\nu)>-1, \qquad (6)$$

$$j_{\alpha,\nu}(z) = \frac{2^\alpha(z/2)^{\nu+\alpha}}{\Gamma(\nu+\alpha+1)}$$
$$\times\,{}_2F_3\left(\begin{matrix}(\nu+1)/2,\ (\nu+2)/2\\ \nu+1,\ (\nu+\alpha+1)/2,\ (\nu+\alpha+2)/2\end{matrix}\middle|\,-z^2/4\right) \qquad (7)$$

If ν is a positive integer r, then $j_{\alpha,r}(z)$ is the rth repeated integral of $J_\nu(z)$

For the development of the above and many other results on integrals of Bessel functions, hypergeometric functions, and related topics, see Luke (1962a)

6 2 11 The Incomplete Gamma Function and Related Functions

Incomplete Gamma Functions

$$\gamma(a,z) = \int_0^z t^{a-1}e^{-t}\,dt = a^{-1}z^a\,{}_1F_1\left(\begin{matrix}a\\ a+1\end{matrix}\middle|\,-z\right)$$
$$= a^{-1}z^a e^{-z}\,{}_1F_1\left(\begin{matrix}1\\ a+1\end{matrix}\middle|\,z\right), \qquad R(a)>0 \qquad (1)$$

$$\Gamma(a, z) = \int_z^{\infty e^{i\delta}} t^{a-1}e^{-t}\,dt = \Gamma(a) - \gamma(a, z) = e^{-z}\psi(1-a; 1-a; z)$$
$$= z^a e^{-z}\psi(1; a+1; z),$$
$$\delta \text{ real}; \qquad |\delta| < \pi/2, \quad R(a) > 0; \qquad |\delta| = \pi/2, \quad 0 < R(a) < 1. \quad (2)$$

Here the path of integration lies in the branch of the cut plane determined by $|\arg z| < \pi$ and is the ray $\eta \exp(i\delta)$, $\eta \to \infty$, except for an initial finite path. If $z \neq 0$, the integral in (2) exists without the restriction on a. If $a \to 0$, we get the exponential integral [see (8)] and in this event, we exclude the origin in the path of integration.

$$\Gamma(a, z) \sim z^{a-1}e^{-z}\,{}_2F_0(1, 1; -1/z),$$
$$|z| \to \infty, \qquad |\arg z| \leqslant 3\pi/2 - \epsilon, \qquad \epsilon > 0. \quad (3)$$

$$\mathrm{Ci}(\alpha, z) + i\,\mathrm{Si}(\alpha, z) = \int_0^z t^{-\alpha}e^{it}\,dt = e^{i\pi(1-\alpha)/2}\gamma(1-\alpha, ze^{-i\pi/2}),$$
$$R(\alpha) < 1. \quad (4)$$

$$\int_0^z t^{-\alpha}\cos t\,dt = \frac{z^{1-\alpha}}{(1-\alpha)}\,{}_1F_2\left(\begin{matrix}\frac{1}{2}(1-\alpha)\\ \frac{1}{2}(3-\alpha), \frac{1}{2}\end{matrix}\,\middle|\,\frac{-z^2}{4}\right), \qquad R(\alpha) < 1. \quad (5)$$

$$\int_0^z t^{-\alpha}\sin t\,dt = \frac{z^{2-\alpha}}{(2-\alpha)}\,{}_1F_2\left(\begin{matrix}\frac{1}{2}(2-\alpha)\\ \frac{1}{2}(4-\alpha), \frac{3}{2}\end{matrix}\,\middle|\,\frac{-z^2}{4}\right), \qquad R(\alpha) < 2. \quad (6)$$

The incomplete gamma functions are special cases of confluent hypergeometric functions. They may also be viewed as a special case of an integral involving Bessel functions. For instance

$$\gamma(a, z) = (2/\pi)^{1/2}\int_0^z t^{a-1/2}K_{1/2}(t)\,dt, \qquad R(a) > 0. \quad (7)$$

For further material on the functions of this section, see Abramowitz and Stegun (1964), Erdélyi *et al.* (1953, Vol. 2, Chapter 9), Luke (1962a), and Nielsen (1965).

The Exponential Integral

$$E_1(z) = -\mathrm{Ei}(-z) = \int_z^{\infty e^{i\delta}} t^{-1}e^{-t}\,dt = \Gamma(0, z) = e^{-z}\psi(1; 1; z). \quad (8)$$

For the path of integration and other remarks, see (2) and the comments following it.

$$\mathrm{Ei}(x) = -\,\mathrm{P.V.}\int_{-x}^{\infty} t^{-1}e^{-t}\,dt$$
$$= \mathrm{P.V.}\int_{-\infty}^{x} t^{-1}e^{t}\,dt = -\Gamma(0, -x) = e^x\psi(1; 1; -x), \quad x > 0. \quad (9)$$

$$\mathrm{Ei}(z) = \mathrm{Ei}(-ze^{\pm i\pi}) \mp i\pi \tag{10}$$

$$\mathrm{Ei}(\pm 2x) = e^{\pm x}(\pi x/2)^{1/2}\left[\frac{\partial I_\nu(x)}{\partial \nu}\bigg|_{\nu=-1/2} \mp \frac{\partial I_\nu(x)}{\partial \nu}\bigg|_{\nu=1/2}\right] \tag{11}$$

$$\int_0^z t^{-1}(1 - e^{-t})\,dt = \lim_{a\to 0}\{a^{-1}z^a - \gamma(a, z)\} \tag{12}$$

$$= \lim_{a\to 0} a^{-1}\left\{1 - e^{-z}\,{}_1F_1\left(\begin{matrix}1\\1+a\end{matrix}\middle| z\right)\right\} \tag{13}$$

$$= \lim_{a\to 0} a^{-1}\left\{1 - {}_1F_1\left(\begin{matrix}a\\1+a\end{matrix}\middle| -z\right)\right\} \tag{14}$$

$$\int_0^z t^{-1}(1 - e^{-t})\,dt = e^{-z}\sum_{n=1}^{\infty}\frac{\{\psi(n+1) - \psi(1)\}\,z^n}{n!} \tag{15}$$

$$\int_0^z t^{-1}(1 - e^{-t})\,dt = z\,{}_2F_2\left(\begin{matrix}1, 1\\2, 2\end{matrix}\middle| -z\right) \tag{16}$$

$$E_1(z) + (\gamma + \ln z) = \int_0^z t^{-1}(1 - e^{-t})\,dt \tag{17}$$

$$\mathrm{Ei}(z) - (\gamma + \ln z) = \int_0^z t^{-1}(e^t - 1)\,dt \tag{18}$$

Cosine and Sine Integrals

$$\mathrm{Ci}(z) + i(\pi/2 - \mathrm{Si}(z)) = -\Gamma(0, ze^{i\pi/2}) = \int_\infty^z t^{-1}e^{-it}\,dt \tag{19}$$

$$\mathrm{Ci}(z) - (\gamma + \ln z) = -\int^z t^{-1}(1 - \cos t)\,dt$$

$$= -(z^2/4)\,{}_2F_3\left(\begin{matrix}1, 1\\2, 2, 3/2\end{matrix}\middle| -z^2/4\right) \tag{20}$$

$$\int_0^z t^{-1}(1 - \cos t)\,dt = \lim_{a\to 0} a^{-1}\{1 - {}_1F_2(a/2, a/2 + 1, 1/2, -z^2/4)\} \tag{21}$$

$$= \lim_{a\to 0} (2a)^{-1}\{1 - {}_1F_2(a, 1, 1/2, -z^2/4)\} \tag{22}$$

$$= \lim_{a\to 0} a^{-1}\left\{1 - \cos z\,{}_1F_2\left(\begin{matrix}1\\(a+1)/2, (a+2)/2\end{matrix}\middle| -z^2/4\right) - \frac{z\sin z}{(1+a)}\,{}_1F_2\left(\begin{matrix}1\\(a+2)/2, (a+3)/2\end{matrix}\middle| -z^2/4\right)\right\} \tag{23}$$

$$\int_0^z t^{-1}\sin t\,dt = \lim_{a\to 0}\left\{\sin z\,{}_1F_2\left(\begin{matrix}1\\(a+1)/2,\,(a+2)/2\end{matrix}\middle|\,-z^2/4\right)\right.$$
$$\left.-\frac{z\cos z}{1+a}\,{}_1F_2\left(\begin{matrix}1\\(a+2)/2,\,(a+3)/2\end{matrix}\middle|\,-z^2/4\right)\right\}. \tag{24}$$

$$\mathrm{si}(z)+\pi/2 = \mathrm{Si}(z) = \int_0^z t^{-1}\sin t\,dt = z\,{}_1F_2\left(\begin{matrix}1/2\\3/2,\,3/2\end{matrix}\middle|\,-z^2/4\right). \tag{25}$$

$$\mathrm{Ci}(2z) = (\pi z/2)^{1/2}\left[\frac{\partial J_\nu(z)}{\partial\nu}\bigg|_{\nu=1/2}\sin z + \frac{\partial J_\nu(z)}{\partial\nu}\bigg|_{\nu=-1/2}\cos z\right]. \tag{26}$$

$$\mathrm{Si}(2z) = (\pi z/2)^{1/2}\left[-\frac{\partial J_\nu(z)}{\partial\nu}\bigg|_{\nu=1/2}\cos z + \frac{\partial J_\nu(z)}{\partial\nu}\bigg|_{\nu=-1/2}\sin z\right]. \tag{27}$$

$$\mathrm{Ci}(z)+i\,\mathrm{si}(z) \sim -z^{-1}e^{iz}\{z^{-1}\,{}_3F_0(3/2,1,1;-4/z^2)$$
$$+\,i\,{}_3F_0(1/2,1,1;-4/z^2)\},$$
$$|z|\to\infty,\qquad |\arg z|\leqslant\pi-\epsilon,\qquad \epsilon>0. \tag{28}$$

Error Functions

$$\mathrm{Erf}(z) = \int_0^z e^{-t^2}\,dt = \tfrac{1}{2}\gamma(\tfrac{1}{2},z^2) = z\,{}_1F_1(\tfrac{1}{2};\tfrac{3}{2};-z^2) = ze^{-z^2}\,{}_1F_1(1;\tfrac{3}{2};z^2). \tag{29}$$

$$\mathrm{Erfc}(z) = \int_z^\infty e^{-t^2}\,dt = \tfrac{1}{2}\pi^{1/2}-\mathrm{Erf}(z) = \tfrac{1}{2}\Gamma(\tfrac{1}{2},z^2) = \tfrac{1}{2}e^{-z^2}\psi(\tfrac{1}{2};\tfrac{1}{2};z^2). \tag{30}$$

The notation $\mathrm{erf}(z) = (2/\pi^{1/2})\,\mathrm{Erf}(z)$ is often used.

$$\mathrm{Erfi}(z) = -i\,\mathrm{Erf}(iz) = \int_0^z e^{t^2}\,dt. \tag{31}$$

$$\mathrm{Erfc}(z) \sim (e^{-z^2}/2z)\,{}_2F_0(1,1/2;-1/z^2)$$
$$|z|\to\infty,\qquad |\arg z|\leqslant 3\pi/4-\epsilon,\qquad \epsilon>0. \tag{32}$$

$$\mathrm{Erfi}(z) \sim (e^{z^2}/2z)\,{}_2F_0(1,1/2;1/z^2)-(i\pi^{1/2}\epsilon/2),$$
$$|z|\to\infty,\qquad -(3+2\epsilon)\,\pi/4<\arg z<(3-2\epsilon)\,\pi/4,\qquad \epsilon=\pm 1. \tag{33}$$

$$i^0\,\mathrm{erfc}(z) = 2\pi^{-1/2}\,\mathrm{Erfc}(z),\qquad i^n\,\mathrm{erfc}(z) = \int_z^\infty i^{n-1}\,\mathrm{erfc}(t)\,dt. \tag{34}$$

$$i^n\,\mathrm{erfc}(z) = \frac{e^{-z^2}}{2^n\Gamma(n/2+1)}\,{}_1F_1\left(\begin{matrix}(n+1)/2\\1/2\end{matrix}\middle|\,z^2\right)$$
$$-\frac{ze^{-z^2}}{2^{n-1}\Gamma[(n+1)/2]}\,{}_1F_1\left(\begin{matrix}n/2+1\\3/2\end{matrix}\middle|\,z^2\right). \tag{35}$$

$$i^n \operatorname{erfc}(z) \sim \frac{2e^{-z^2}}{\pi^{1/2}(2z)^{n+1}}\, {}_2F_0((n+1)/2,\ n/2+1;\ -1/z^2)$$
$$|z| \to \infty \qquad |\arg z| \leqslant 3\pi/4 - \epsilon \qquad \epsilon > 0 \tag{36}$$

$$i^n \operatorname{erfc}(z) = \frac{e^{-z^2}\, {}_2e^{-z^2/2n^{1/2}}}{2^n\Gamma(n/2+1)}\,[1 + O(n^{-1/2})] \qquad z \text{ bounded} \quad n \to \infty \tag{37}$$

$$i^n \operatorname{erfc}(z) + (z/n)\, i^{n-1} \operatorname{erfc}(z) - (2n)^{-1}\, i^{n-2} \operatorname{erfc}(z) = 0 \tag{38}$$

Fresnel Integrals

$$C(z) + iS(z) = (2\pi)^{-1/2} \int_0^z t^{-1/2} e^{it}\, dt = (2\pi)^{-1/2} e^{i\pi/4} \gamma(\tfrac{1}{2};\ ze^{-i\pi/2})$$
$$= 2(2\pi)^{-1/2} e^{i\pi/4} \operatorname{Erf}(z^{1/2} e^{-i\pi/4}) \tag{39}$$

$$C(z) + iS(z) = (2\pi)^{-1/2} e^{i\pi/4}\{\pi^{1/2} - \Gamma(\tfrac{1}{2};\ ze^{-i\pi/2})\} \tag{40}$$

$$C(z) + iS(z) = (2z/\pi)^{1/2} \left\{ {}_1F_2\left(\begin{matrix}\frac{1}{4} \\ \frac{5}{4},\ \frac{1}{2}\end{matrix}\middle|\, \frac{-z^2}{4}\right) + \tfrac{1}{3}(iz)\, {}_1F_2\left(\begin{matrix}\frac{3}{4} \\ \frac{7}{4},\ \frac{3}{2}\end{matrix}\middle|\, \frac{-z^2}{4}\right)\right\} \tag{41}$$

$$C(z) + iS(z) = (2z/\pi)^{1/2} e^{iz} \left\{ {}_1F_2\left(\begin{matrix}1 \\ \frac{3}{4},\ \frac{5}{4}\end{matrix}\middle|\, \frac{-z^2}{4}\right) - \tfrac{2}{3}(iz)\, {}_1F_2\left(\begin{matrix}1 \\ \frac{5}{4},\ \frac{7}{4}\end{matrix}\middle|\, \frac{-z^2}{4}\right)\right\} \tag{42}$$

$$C(z) + iS(z) \sim (1+i)/2 - (2\pi z)^{-1/2} e^{iz}\{(2z)^{-1}\, {}_3F_0(1,\ 3/4,\ 5/4;\ -4/z^2)$$
$$+ i\, {}_3F_0(1,\ 1/4,\ 3/4;\ -4/z^2)\} \qquad |z| \to \infty \qquad |\arg z| < \pi \tag{43}$$

6.3 The ${}_pF_q$ Expressed as a Named Function

Function	Location
${}_0F_0(z) = e^z$	
${}_1F_0(a;\ z) = (1+z)^{-a}$, $\lvert z\rvert < 1$	
${}_0F_1(;\ 1+\nu;\ z^2/4)$	6.2.1(9, 10), 6.2.7(1, 2), 6.2.8
${}_1F_1(a;\ c;\ z)$	Chapter IV, 6.2.6(1, 2, 5), 6.2.7(3, 4), 6.2.11(1, 13, 14, 29, 35), 8.1(33)
${}_2F_1(a, b;\ c;\ z)$	Chapter III, 6.2.1(3, 8, 11, 18), 6.2.3, 6.2.5, 8.1(25)
${}_2F_0(a, b;\ z)$	4.7(1), 6.2.7(32, 33, 37, 38), 6.2.11(3, 32, 33, 36)
${}_1F_2\left(\begin{matrix}a \\ b,\ c\end{matrix}\middle|\, \right)$	6.2.7(40–42), 6.2.9(1, 3, 5), 6.2.10(1), 6.2.11(5, 6, 21, 25, 39, 40)
${}_2F_2\left(\begin{matrix}a,\ b \\ c,\ d\end{matrix}\middle|\, z\right)$	6.2.9(25), 6.2.10(3), 6.2.11(16)
${}_3F_3\left(\begin{matrix}a,\ b,\ c \\ d,\ e\end{matrix}\middle|\, z\right)$	3.13.3

Function	Location
${}_3F_1\left(\begin{matrix}a, b, c\\ d\end{matrix}\middle\vert z\right)$	6.2.9(28)
${}_3F_0(a, b, c; z)$	6.2.9(22–24), 6.2.11(28, 41)
${}_0F_3(; a, b, c \mid z)$	6.2.7(43, 44)
${}_2F_3\left(\begin{matrix}a, b\\ c, d, e\end{matrix}\middle\vert z\right)$	6.2.6(1), 6.2.7(39), 6.2.10(5, 7), 6.2.11(20), 10.4(29, 30)
${}_4F_3\left(\begin{matrix}a, b, c, d\\ e, f, g\end{matrix}\middle\vert z\right)$	3.13.3(34, 46–49, 56, 57)
${}_4F_1\left(\begin{matrix}a, b, c, d\\ e\end{matrix}\middle\vert z\right)$	6.2.7(36), 16.1(29)

6.4. Named Functions Expressed in Terms of the G-Function

$$ {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle\vert z\right) = \frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)}\, G^{1,p}_{p,q+1}\left(-z\,\middle|\,\begin{matrix}1-\alpha_p\\ 0, 1-\rho_q\end{matrix}\right), $$

$$ p \leqslant q \quad \text{or} \quad p = q+1 \quad \text{and} \quad |z| < 1. \tag{1} $$

$$ {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle\vert z\right) = \frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)}\, G^{p,1}_{q+1,p}\left(-z^{-1}\,\middle|\,\begin{matrix}1, \rho_q\\ \alpha_p\end{matrix}\right), $$

$$ p \leqslant q \quad \text{or} \quad p = q+1 \quad \text{and} \quad |z| < 1. \tag{2} $$

$$ \begin{aligned} & z^a e^{z/2} M_{k,m}(z) \\ &= z^{1/2}\Gamma(2m+1)\,\Gamma(k+\tfrac{1}{2}-m)\, G^{1,0}_{1,2}\left(z\,\middle|\,\begin{matrix}a+\frac{1}{2}+k\\ a+m, a-m\end{matrix}\right) \\ &= \frac{(2\pi z)^{1/2}\,\Gamma(2m+1)\,\Gamma(k+\frac{1}{2}-m)}{2^{k-a}} \\ &\quad \times G^{2,0}_{2,4}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\frac{a}{2}+\frac{1}{4}+\frac{k}{2}, \frac{a}{2}+\frac{3}{4}+\frac{k}{2}\\ \frac{a+m}{2}, \frac{a+m+1}{2}, \frac{a-m}{2}, \frac{a-m+1}{2}\end{matrix}\right). \end{aligned} \tag{3} $$

$$ \begin{aligned} & z^a e^{-z/2} M_{k,m}(z) \\ &= \frac{z^{1/2}\Gamma(2m+1)}{\Gamma(k+\frac{1}{2}+m)}\, G^{1,1}_{1,2}\left(z\,\middle|\,\begin{matrix}a+\frac{1}{2}-k\\ a+m, a-m\end{matrix}\right) \\ &= \frac{(z/2\pi)^{1/2}\, 2^{k+a}\Gamma(2m+1)}{\Gamma(k+\frac{1}{2}+m)} \\ &\quad \times G^{2,2}_{2,4}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\frac{a}{2}+\frac{1}{4}-\frac{k}{2}, \frac{a}{2}+\frac{3}{4}-\frac{k}{2}\\ \frac{a+m}{2}, \frac{a+m+1}{2}, \frac{a-m}{2}, \frac{a-m+1}{2}\end{matrix}\right). \end{aligned} \tag{4} $$

$$z^a e^{z/2} W_{k,m}(z)$$
$$= [\Gamma(\tfrac{1}{2}-k+m)\,\Gamma(\tfrac{1}{2}-k-m)]^{-1}\, G^{2\,1}_{1\,2}\left(z \,\middle|\, \begin{matrix} a+k+1 \\ a+\frac{1}{2}+m,\, a+\frac{1}{2}-m \end{matrix}\right)$$
$$= \frac{z^{1/2} 2^{a-k}}{(2\pi)^{3/2}\,\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}$$
$$\times G^{4\,2}_{2\,4}\left(\frac{z^2}{4}\,\middle|\, \begin{matrix} \frac{a}{2}+\frac{1}{4}+\frac{k}{2},\, \frac{a}{2}+\frac{3}{4}+\frac{k}{2} \\ \frac{a+m}{2},\, \frac{a+m+1}{2},\, \frac{a-m}{2},\, \frac{a-m+1}{2} \end{matrix}\right). \tag{5}$$

$$z^a e^{-z/2} W_{k,m}(z)$$
$$= z^{1/2}\, G^{2\,0}_{1\,2}\left(z \,\middle|\, \begin{matrix} a+\frac{1}{2}-k \\ a+m,\, a-m \end{matrix}\right)$$
$$= (z/2\pi)^{1/2}\, 2^{a+k}$$
$$\times G^{4\,0}_{2\,4}\left(\frac{z^2}{4}\,\middle|\, \begin{matrix} \frac{a}{2}+\frac{1}{4}-\frac{k}{2},\, \frac{a}{2}+\frac{3}{4}-\frac{k}{2} \\ \frac{a+m}{2},\, \frac{a+m+1}{2},\, \frac{a-m}{2},\, \frac{a-m+1}{2} \end{matrix}\right) \tag{6}$$

$$z^\mu J_\nu(z) = 2^\mu\, G^{1\,0}_{0\,2}\left(\frac{z^2}{4}\,\middle|\, \frac{\mu+\nu}{2},\, \frac{\mu-\nu}{2}\right) \tag{7}$$

$$z^\mu J_\nu(z) = 2^{2\mu}\, G^{2\,0}_{0\,4}\left(\frac{z^4}{256}\,\middle|\, \frac{\mu+\nu}{4},\, \frac{\mu+\nu+2}{4},\, \frac{\mu-\nu}{4},\, \frac{\mu-\nu+2}{4}\right) \tag{8}$$

$$\pi^{1/2} e^{-z} (2z)^\mu I_\nu(z) = G^{1\,1}_{1\,2}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2}+\mu \\ \mu+\nu,\, \mu-\nu \end{matrix}\right) \tag{9}$$

$$z^\mu Y_\nu(z) = (-)^m\, 2^\mu\, G^{2\,0}_{1\,3}\left(\frac{z^2}{4}\,\middle|\, \begin{matrix} \frac{\mu-\nu-1}{2}-m \\ \frac{\mu+\nu}{2},\, \frac{\mu-\nu}{2},\, \frac{\mu-\nu-1}{2}-m \end{matrix}\right),$$
$$m = 0, \pm 1, \pm 2, \quad . \tag{10}$$

$$z^\mu K_\nu(z) = 2^{\mu-1}\, G^{2\,0}_{0\,2}\left(\frac{z^2}{4}\,\middle|\, \frac{\mu+\nu}{2},\, \frac{\mu-\nu}{2}\right) \tag{11}$$

$$e^{-z}(2z)^\mu K_\nu(z) = \pi^{1/2}\, G^{2\,0}_{1\,2}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2}+\mu \\ \mu+\nu,\, \mu-\nu \end{matrix}\right) \tag{12}$$

$$e^{z}(2z)^\mu K_\nu(z) = \pi^{-1/2} \cos \nu\pi\, G^{2\,1}_{1\,2}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2}+\mu \\ \mu+\nu,\, \mu-\nu \end{matrix}\right) \tag{13}$$

$$(z/2)^{\omega} s_{\mu,\nu}(z) = 2^{\mu-1}\Gamma[(\mu+\nu+1)/2]\,\Gamma[(\mu-\nu+1)/2] \times G^{1,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\dfrac{\mu+\omega+1}{2}\\ \dfrac{\mu+\omega+1}{2}, \dfrac{\omega+\nu}{2}, \dfrac{\omega-\nu}{2}\end{matrix}\right). \tag{14}$$

$$(z/2)^{\omega}\,\mathbf{H}_{\nu}(z) = G^{1,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\dfrac{\nu+\omega+1}{2}\\ \dfrac{\nu+\omega+1}{2}, \dfrac{\omega+\nu}{2}, \dfrac{\omega-\nu}{2}\end{matrix}\right). \tag{15}$$

$$(z/2)^{\omega} S_{\mu,\nu}(z) = \frac{2^{\mu-1}}{\Gamma[(1-\mu+\nu)/2]\,\Gamma[(1-\mu-\nu)/2]} \times G^{3,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\dfrac{\mu+\omega+1}{2}\\ \dfrac{\mu+\omega+1}{2}, \dfrac{\omega+\nu}{2}, \dfrac{\omega-\nu}{2}\end{matrix}\right). \tag{16}$$

$$(z/2)^{\omega}[\mathbf{H}_{\nu}(z) - Y_{\nu}(z)] = \frac{\cos\nu\pi}{\pi^2}\, G^{3,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\dfrac{\nu+\omega+1}{2}\\ \dfrac{\nu+\omega+1}{2}, \dfrac{\omega+\nu}{2}, \dfrac{\omega-\nu}{2}\end{matrix}\right). \tag{17}$$

$$(z/2)^{\omega}[I_{\nu}(z) - \mathbf{L}_{\nu}(z)] = \pi^{-1}\, G^{2,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\dfrac{\nu+\omega+1}{2}\\ \dfrac{\nu+\omega+1}{2}, \dfrac{\omega+\nu}{2}, \dfrac{\omega-\nu}{2}\end{matrix}\right). \tag{18}$$

$$(z/2)^{\omega}[I_{-\nu}(z) - \mathbf{L}_{\nu}(z)] = \pi^{-1}\cos\nu\pi\, G^{2,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\dfrac{\nu+\omega+1}{2}\\ \dfrac{\nu+\omega+1}{2}, \dfrac{\omega-\nu}{2}, \dfrac{\omega+\nu}{2}\end{matrix}\right). \tag{19}$$

$$\mathrm{Ci}(z) + i\,\mathrm{si}(z) = -\frac{z^{-1}e^{iz}}{\pi^{1/2}}\left\{2z^{-1}\, G^{3,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}1\\ \frac{3}{2}, 1, 1\end{matrix}\right) + i\, G^{3,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}1\\ \frac{1}{2}, 1, 1\end{matrix}\right)\right\}. \tag{20}$$

$$C(z) + iS(z) = \tfrac{1}{2}(1+i) - \frac{(2\pi z)^{-1/2}\, e^{iz}}{2^{1/2}\pi}\left\{\left(\frac{2}{z}\right) G^{3,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}1\\ \frac{3}{4}, \frac{5}{4}, 1\end{matrix}\right) + i\, G^{3,1}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}1\\ \frac{1}{4}, \frac{3}{4}, 1\end{matrix}\right)\right\}. \tag{21}$$

$$z^a M_{k,m}(iz)\, M_{k,m}(-iz)$$

$$= \frac{2^{a-1}\pi^{1/2}[\Gamma(2m+1)]}{\Gamma(\frac{1}{2}+k+m)\,\Gamma(\frac{1}{2}-k+m)}$$

$$\times G^{1,2}_{2,4}\left(\frac{z^2}{4}\middle|\begin{matrix} \frac{a}{2}+1+k & \frac{a}{2}+1-k \\ \frac{a+1}{2}+m & \frac{a+1}{2}-m & \frac{a}{2}+1 & \frac{a+1}{2} \end{matrix}\right) \tag{22}$$

$$z^a M_{k,m}(z)\, W_{-k,m}(z)$$

$$= \frac{2^a\Gamma(2m+1)}{\pi^{1/2}\Gamma(\frac{1}{2}+k+m)}$$

$$\times G^{3,1}_{2,4}\left(\frac{z^2}{4}\middle|\begin{matrix} \frac{a}{2}+1-k & \frac{a}{2}+1+k \\ \frac{a+1}{2}+m & \frac{a}{2}+1 & \frac{a+1}{2} & \frac{a+1}{2}-m \end{matrix}\right) \tag{23}$$

$$z^a W_{k,m}(z)\, W_{-k,m}(z)$$

$$= \frac{2^a}{\pi^{1/2}}\, G^{4,0}_{2,4}\left(\frac{z^2}{4}\middle|\begin{matrix} \frac{a}{2}+1+k & \frac{a}{2}+1-k \\ \frac{a+1}{2}+m & \frac{a+1}{2}-m & \frac{a}{2}+1 & \frac{a+1}{2} \end{matrix}\right) \tag{24}$$

$$z^a W_{k,m}(iz)\, W_{k,m}(-iz)$$

$$= \frac{2^a}{\pi^{1/2}\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}$$

$$\times G^{4,1}_{2,4}\left(\frac{z^2}{4}\middle|\begin{matrix} \frac{a}{2}+1+k & \frac{a}{2}+1-k \\ \frac{a+1}{2}+m & \frac{a+1}{2}-m & \frac{a}{2}+1 & \frac{a+1}{2} \end{matrix}\right) \tag{25}$$

$$z^\omega J_\mu(z)\, J_\nu(z)$$

$$= \pi^{-1/2}\, G^{1,2}_{2,4}\left(z^2\middle|\begin{matrix} \frac{\omega+1}{2} & \frac{\omega}{2} \\ \frac{\omega+\mu+\nu}{2} & \frac{\omega-\mu+\nu}{2} & \frac{\omega+\mu-\nu}{2} & \frac{\omega-\mu-\nu}{2} \end{matrix}\right) \tag{26}$$

$$z^\omega J_\nu^2(z) = \pi^{-1/2}\, G^{1,1}_{1,3}\left(z^2\middle|\begin{matrix} \frac{\omega+1}{2} \\ \frac{\omega}{2}+\nu & \frac{\omega}{2} & -\nu \end{matrix}\right) \tag{27}$$

$$z^{\omega} J_{-\nu}(z)\, J_{\nu}(z) = \pi^{-1/2}\, G^{1,1}_{1,3}\left(z^2 \left| \begin{matrix} \dfrac{\omega+1}{2} \\ \dfrac{\omega}{2}, \dfrac{\omega}{2}+\nu, \dfrac{\omega}{2}-\nu \end{matrix} \right. \right). \tag{28}$$

$$z^{\omega} J_{\nu}(z)\, I_{\nu}(z) = \pi^{1/2} 2^{3\omega/2}\, G^{1,0}_{0,4}\left(\frac{z^4}{64} \left| \frac{\omega+2\nu}{4}, \frac{\omega-2\nu}{4}, \frac{\omega}{4}, \frac{\omega+2}{4} \right. \right). \tag{29}$$

$$z^{\omega} J_{\nu}(z)\, Y_{\nu}(z) = -\pi^{-1/2}\, G^{2,0}_{1,3}\left(z^2 \left| \begin{matrix} \dfrac{\omega+1}{2} \\ \dfrac{\omega}{2}, \dfrac{\omega}{2}+\nu, \dfrac{\omega}{2}-\nu \end{matrix} \right. \right). \tag{30}$$

$$z^{\omega} I_{\nu}(z)\, K_{\mu}(z)$$
$$= (4\pi)^{-1/2}\, G^{2,2}_{2,4}\left(z^2 \left| \begin{matrix} \dfrac{\omega}{2}, \dfrac{\omega+1}{2} \\ \dfrac{\omega+\mu+\nu}{2}, \dfrac{\omega-\mu+\nu}{2}, \dfrac{\omega+\mu-\nu}{2}, \dfrac{\omega-\mu-\nu}{2} \end{matrix} \right. \right). \tag{31}$$

$$z^{\omega} J_{\nu}(z)\, K_{\nu}(z) = \pi^{-1/2} 2^{3\omega/2-1/2}\, G^{3,0}_{0,4}\left(\frac{z^4}{64} \left| \frac{\omega+2\nu}{4}, \frac{\omega+2}{4}, \frac{\omega}{4}, \frac{\omega-2\nu}{4} \right. \right). \tag{32}$$

$$z^{\omega} K_{\mu}(z)\, K_{\nu}(z)$$
$$= 2^{-1}\pi^{1/2}\, G^{4,0}_{2,4}\left(z^2 \left| \begin{matrix} \dfrac{\omega}{2}, \dfrac{\omega+1}{2} \\ \dfrac{\omega+\mu+\nu}{2}, \dfrac{\omega-\mu+\nu}{2}, \dfrac{\omega+\mu-\nu}{2}, \dfrac{\omega-\mu-\nu}{2} \end{matrix} \right. \right). \tag{33}$$

$$z^{\omega} H^{(1)}_{\nu}(z)\, H^{(2)}_{\nu}(z) = \pi^{-5/2} 2 \cos \nu\pi\, G^{3,1}_{1,3}\left(z^2 \left| \begin{matrix} \dfrac{\omega+1}{2} \\ \dfrac{\omega}{2}+\nu, \dfrac{\omega}{2}-\nu, \dfrac{\omega}{2} \end{matrix} \right. \right). \tag{34}$$

$$z^{\omega}[J_{\mu}(z)\, J_{\nu}(z) + J_{-\mu}(z)\, J_{-\nu}(z)]$$
$$= \{[2 \cos(\mu+\nu)\pi/2]/\pi^{1/2}\}$$
$$\times\, G^{2,1}_{2,4}\left(z^2 \left| \begin{matrix} \dfrac{\omega+1}{2}, \dfrac{\omega}{2} \\ \dfrac{\omega+\mu+\nu}{2}, \dfrac{\omega-\mu-\nu}{2}, \dfrac{\omega-\mu+\nu}{2}, \dfrac{\omega+\mu-\nu}{2} \end{matrix} \right. \right). \tag{35}$$

$$z^{\omega}[J_{\mu}(z)\,J_{\nu}(z) - J_{-\mu}(z)\,J_{-\nu}(z)]$$
$$= -\{[2\sin(\mu+\nu)\pi/2]/\pi^{3/2}\}$$
$$\times G^{2\,1}_{2\,4}\left(z^2\left|\begin{matrix}\frac{\omega}{2}, \frac{\omega+1}{2}\\ \frac{\omega+\mu+\nu}{2}, \frac{\omega-\mu-\nu}{2}, \frac{\omega-\mu+\nu}{2}, \frac{\omega+\mu-\nu}{2}\end{matrix}\right.\right) \tag{36}$$

$$z^{\omega}[H^{(1)}_{\mu}(z)\,H^{(1)}_{\nu}(z) - H^{(2)}_{\mu}(z)\,H^{(2)}_{\nu}(z)]$$
$$= \frac{4}{i\pi^{1/2}}\,G^{2\,0}_{2\,4}\left(z^2\left|\begin{matrix}\frac{\omega}{2}, \frac{\omega+1}{2}\\ \frac{\omega+\mu+\nu}{2}, \frac{\omega-\mu+\nu}{2}, \frac{\omega+\mu-\nu}{2}, \frac{\omega-\mu-\nu}{2}\end{matrix}\right.\right) \tag{37}$$

$$z^{\omega}[I_{\mu}(z)\,I_{\nu}(z) - I_{-\mu}(z)\,I_{-\nu}(z)]$$
$$= -\frac{\sin(\mu+\nu)\pi}{\pi^{3/2}}$$
$$\times G^{2\,2}_{2\,4}\left(z^2\left|\begin{matrix}\frac{\omega}{2}, \frac{\omega+1}{2}\\ \frac{\omega+\mu+\nu}{2}, \frac{\omega-\mu-\nu}{2}, \frac{\omega-\mu+\nu}{2}, \frac{\omega+\mu-\nu}{2}\end{matrix}\right.\right) \tag{38}$$

$$(z^{\omega}/2i)[e^{i\pi(\nu-\mu)/2}H^{(1)}_{\nu}(z)\,H^{(2)}_{\mu}(z) - e^{i\pi(\mu-\nu)/2}H^{(1)}_{\mu}(z)\,H^{(2)}_{\nu}(z)]$$
$$= \pi^{-5/2}(\cos\mu\pi - \cos\nu\pi)$$
$$\times G^{4\,1}_{2\,4}\left(z^2\left|\begin{matrix}\frac{\omega}{2}, \frac{\omega+1}{2}\\ \frac{\omega+\mu+\nu}{2}, \frac{\omega-\mu+\nu}{2}, \frac{\omega+\mu-\nu}{2}, \frac{\omega-\mu-\nu}{2}\end{matrix}\right.\right) \tag{39}$$

$$\tfrac{1}{2}(z^{\omega})[e^{i\pi(\nu-\mu)/2}H^{(1)}_{\nu}(z)\,H^{(2)}_{\mu}(z) + e^{i\pi(\mu-\nu)/2}H^{(1)}_{\mu}(z)\,H^{(2)}_{\nu}(z)]$$
$$= \pi^{-5/2}(\cos\mu\pi + \cos\nu\pi)$$
$$\times G^{4\,1}_{2\,4}\left(z^2\left|\begin{matrix}\frac{\omega+1}{2}, \frac{\omega}{2}\\ \frac{\omega+\mu+\nu}{2}, \frac{\omega-\mu+\nu}{2}, \frac{\omega+\mu-\nu}{2}, \frac{\omega-\mu-\nu}{2}\end{matrix}\right.\right) \tag{40}$$

6.5. The G-Function Expressed as a Named Function

$$G^{m\,n}_{p\,q}\left(z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right) = \sum_{h=1}^{m}\frac{\prod_{j=1}^{m}\Gamma(b_j-b_h)^*\prod_{j=1}^{n}\Gamma(1+b_h-a_j)\,z^{b_h}}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{j=n+1}^{p}\Gamma(a_j-b_h)}$$
$$\times\,{}_pF_{q-1}\left(\begin{matrix}1+b_h-a_p\\ 1+b_h-b_q^*\end{matrix}\middle|(-)^{p-m-n}z\right),$$
$$p<q \quad \text{or} \quad p=q \quad \text{and} \quad |z|<1 \tag{1}$$

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)=\sum_{h=1}^{n}\frac{\prod_{j=1}^{n}\Gamma(a_h-a_j)^*\prod_{j=1}^{m}\Gamma(b_j-a_h+1)\,z^{a_h-1}}{\prod_{j=n+1}^{p}\Gamma(1+a_j-a_h)\prod_{j=m+1}^{q}\Gamma(a_h-b_j)}$$

$$\times\,{}_qF_{p-1}\left(\begin{matrix}1+b_q-a_h\\ 1+a_p-a_h^*\end{matrix}\,\middle|\,\frac{(-)^{q-m-n}}{z}\right),$$

$$q<p\quad\text{or}\quad q=p\quad\text{and}\quad |z|>1. \tag{2}$$

$$G_{p,q}^{1,n}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)=\frac{\prod_{j=1}^{n}\Gamma(1+b_1-a_j)\,z^{b_1}}{\prod_{j=2}^{q}\Gamma(1+b_1-b_j)\prod_{j=n+1}^{p}\Gamma(a_j-b_1)}$$

$$\times\,{}_pF_{q-1}\left(\begin{matrix}1+b_1-a_p\\ 1+b_1-b_q^*\end{matrix}\,\middle|\,(-)^{p-1-n}\,z\right),$$

$$p<q\quad\text{or}\quad p=q\quad\text{and}\quad |z|<1. \tag{3}$$

$$G_{p,q}^{m,1}\left(z\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)=\frac{\prod_{j=1}^{m}\Gamma(b_j-a_1+1)\,z^{a_1-1}}{\prod_{j=2}^{p}\Gamma(1+a_j-a_1)\prod_{j=m+1}^{q}\Gamma(a_1-b_j)}$$

$$\times\,{}_qF_{p-1}\left(\begin{matrix}1+b_q-a_1\\ 1+a_p-a_1^*\end{matrix}\,\middle|\,\frac{(-)^{q-m-1}}{z}\right),$$

$$q<p\quad\text{or}\quad q=p\quad\text{and}\quad |z|>1. \tag{4}$$

$$G_{1,2}^{1,1}\left(z\,\middle|\,\begin{matrix}1-k\\ \frac{1}{2}+m,\ \frac{1}{2}-m\end{matrix}\right)=\frac{e^{-z/2}\Gamma(\frac{1}{2}+k+m)}{\Gamma(2m+1)}\,M_{k,m}(z). \tag{5}$$

$$G_{0,2}^{1,0}(z\mid a,b)=z^{\frac{1}{2}(a+b)}J_{a-b}(2z^{1/2}). \tag{6}$$

$$G_{1,2}^{1,0}\left(z\,\middle|\,\begin{matrix}\frac{1}{2}\\ a,\ -a\end{matrix}\right)=\pi^{-1/2}(\cos a\pi)\,e^{z/2}I_a(z/2). \tag{7}$$

$$G_{0,2}^{2,0}(z\mid a,b)=2z^{\frac{1}{2}(a+b)}K_{a-b}(2z^{1/2}). \tag{8}$$

$$G_{1,2}^{2,0}\left(z\,\middle|\,\begin{matrix}\frac{1}{2}\\ b,\ -b\end{matrix}\right)=\pi^{-1/2}e^{-z/2}K_b(z/2). \tag{9}$$

$$G_{1,2}^{2,1}\left(z\,\middle|\,\begin{matrix}\frac{1}{2}\\ b,\ -b\end{matrix}\right)=\frac{\pi^{1/2}e^{z/2}}{\cos b\pi}\,K_b(z/2). \tag{10}$$

$$G_{1,2}^{2,0}\left(z\,\middle|\,\begin{matrix}a\\ b,\ c\end{matrix}\right)=z^{\frac{1}{2}(b+c-1)}e^{-z/2}W_{\frac{1}{2}(1+b+c-2a),\,\frac{1}{2}(b-c)}(z). \tag{11}$$

$$G_{1,2}^{2,1}\left(z\,\middle|\,\begin{matrix}a\\ b,\ c\end{matrix}\right)=\Gamma(b-a+1)\,\Gamma(c-a+1)$$

$$\times\,z^{\frac{1}{2}(b+c-1)}e^{z/2}W_{\frac{1}{2}(2a-b-c-1),\,\frac{1}{2}(b-c)}(z). \tag{12}$$

$$G^{1\,0}_{0\,4}(z \mid a+b/2 \;\; a \;\; a-b/2 \;\; a+1/2) = \pi^{-1/2} z^{a} J_b(x) I_b(x)$$

$$x = 2^{3/2} z^{1/4} \tag{13}$$

$$G^{1\,0}_{0\,4}(z \mid a+1/2 \;\; a \;\; a+b/2 \;\; a-b/2) = \frac{\pi^{-1/2} z^{a}}{2 \sin b\pi/2} \times [J_b(x) I_{-b}(x) - J_{-b}(x) I_b(x)]$$

$$x = 2^{3/2} z^{1/4} \tag{14}$$

$$G^{2\,0}_{0\,4}(z \mid a \;\; a+\tfrac{1}{2} \;\; b \;\; b+\tfrac{1}{2}) = z^{\frac{1}{2}a+\frac{1}{2}b} J_{2a-b}(4z^{1/4}) \tag{15}$$

$$G^{2\,0}_{0\,4}(z \mid a \;\; a \;\; 0 \;\; \tfrac{1}{2}) = \frac{-\pi^{1/2}}{\sin 2\pi a} [J_{2a}(xe^{i\pi/4}) J_{2a}(xe^{-i\pi/4}) - J_{-2a}(xe^{i\pi/4}) J_{-2a}(xe^{-i\pi/4})]$$

$$x = 2^{3/2} z^{1/4} \tag{16}$$

$$G^{2\,0}_{0\,4}(z \mid 0 \;\; \tfrac{1}{2} \;\; a \;\; -a) = \frac{\pi^{1/2}}{i \sin 2\pi a} \times [e^{2i\pi a} J_{2a}(xe^{-i\pi/4}) J_{-2a}(xe^{i\pi/4}) - e^{-2i\pi a} J_{2a}(xe^{i\pi/4}) J_{-2a}(xe^{-i\pi/4})]$$

$$x = 2^{3/2} z^{1/4} \tag{17}$$

$$G^{3\,0}_{0\,4}(z \mid 3a \;\; \tfrac{1}{2} \;\; a \;\; -a-\tfrac{1}{2} \;\; a-\tfrac{1}{2}) = \frac{2\pi^{1/2} z^{a-1/2}}{\cos 2\pi a} K_{4a}(x)[J_{4a}(x) + J_{-4a}(x)]$$

$$x = 2^{3/2} z^{1/4} \tag{18}$$

$$G^{3\,0}_{0\,4}(z \mid 0 \;\; a \;\; \tfrac{1}{2} \;\; -a-\tfrac{1}{2} \;\; \tfrac{1}{2}) = 4(\pi/z)^{1/2} K_{2a}(x) \times [\cos a\pi \, J_{2a}(x) - \sin a\pi \, Y_{2a}(x)]$$

$$x = 2^{3/2} z^{1/4} \tag{19}$$

$$G^{3\,0}_{0\,4}(z \mid -\tfrac{1}{2} \;\; a-\tfrac{1}{2} \;\; -a-\tfrac{1}{2} \;\; 0) = -4(\pi/z)^{1/2} K_{2a}(x) \times [\sin \pi a \, J_{2a}(x) + \cos \pi a \, Y_{2a}(x)]$$

$$x = 2^{3/2} z^{1/4} \tag{20}$$

$$G^{3\,0}_{0\,4}(z \mid a \;\; b+\tfrac{1}{2} \;\; b \;\; 2b-a) = (2\pi)^{1/2} z^{b} K_{2a-b}(x) J_{2a-b}(x)$$

$$x = 2^{3/2} z^{1/4} \tag{21}$$

$$G_{0,4}^{4,0}(z \mid a, a+\tfrac{1}{2}, b, b+\tfrac{1}{2}) = 4\pi z^{\frac{1}{2}(a+b)} K_{2(a-b)}(4z^{1/4}). \tag{22}$$

$$G_{0,4}^{4,0}(z \mid a, a+\tfrac{1}{2}, b, 2a-b) = 8\pi^{1/2} z^{a} K_{2(b-a)}(xe^{i\pi/4})\, K_{2(b-a)}(xe^{-i\pi/4}),$$

$$x = 2^{3/2} z^{1/4}. \tag{23}$$

$$G_{2,4}^{4,0}\left(z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+k, \frac{1}{2}-k \\ 0, \frac{1}{2}, m, -m \end{matrix}\right) = \pi^{1/2} z^{-1} W_{k,m}(2z)\, W_{-k,m}(2z). \tag{24}$$

$$G_{2,4}^{4,1}\left(z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+k, \frac{1}{2}-k \\ 0, \frac{1}{2}, m, -m \end{matrix}\right) = \pi^{1/2} z^{-1} \Gamma(\tfrac{1}{2}+m-k)\, \Gamma(\tfrac{1}{2}-m-k)$$
$$\times\, W_{k,m}(2iz)\, W_{k,m}(-2iz). \tag{25}$$

$$G_{1,3}^{1,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, a, -a \end{matrix}\right) = \pi^{1/2} J_a(z^{1/2})\, J_{-a}(z^{1/2}). \tag{26}$$

$$G_{1,3}^{1,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ a, 0, -a \end{matrix}\right) = \pi^{1/2} J_a^2(z^{1/2}). \tag{27}$$

$$G_{1,3}^{1,1}\left(z \,\middle|\, \begin{matrix} a \\ a, b, a-\frac{1}{2} \end{matrix}\right) = z^{(2a+2b-1)/4} \mathbf{H}_{a-b-1/2}(2z^{1/2}). \tag{28}$$

$$G_{1,3}^{2,0}\left(z \,\middle|\, \begin{matrix} a-\frac{1}{2} \\ a, b, a-\frac{1}{2} \end{matrix}\right) = z^{(a+b)/2} Y_{b-a}(2z^{1/2}). \tag{29}$$

$$G_{1,3}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ -a, a, 0 \end{matrix}\right) = \frac{\pi^{1/2}}{2\sin a\pi}\, [J_{-a}^2(z^{1/2}) - J_a^2(z^{1/2})]. \tag{30}$$

$$G_{1,3}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, a, -a \end{matrix}\right) = -\pi^{1/2} J_a(z^{1/2})\, Y_a(z^{1/2}). \tag{31}$$

$$G_{1,3}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, a, -a \end{matrix}\right) = 2\pi^{1/2} I_a(z^{1/2})\, K_a(z^{1/2}). \tag{32}$$

$$G_{1,3}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ a, -a, 0 \end{matrix}\right) = \frac{\pi^{3/2}}{\sin 2\pi a}\, [I_{-a}^2(z^{1/2}) - I_a^2(z^{1/2})]. \tag{33}$$

$$G_{1,3}^{2,1}\left(z \,\middle|\, \begin{matrix} a+\frac{1}{2} \\ a+\frac{1}{2}, b, a \end{matrix}\right) = \frac{\pi z^{(a+b)/2}}{\cos(b-a)\pi}\, [I_{b-a}(2z^{1/2}) - \mathbf{L}_{a-b}(2z^{1/2})]. \tag{34}$$

$$G_{1,3}^{2,1}\left(z \,\middle|\, \begin{matrix} a+\frac{1}{2} \\ a, a+\frac{1}{2}, b \end{matrix}\right) = \pi z^{(a+b)/2} [I_{a-b}(2z^{1/2}) - \mathbf{L}_{a-b}(2z^{1/2})]. \tag{35}$$

$$G_{1,3}^{3,0}\left(z \,\middle|\, \begin{matrix} a+\frac{1}{2} \\ a+b, a-b, a \end{matrix}\right) = 2\pi^{-1/2} z^{a} K_b^2(z^{1/2}). \tag{36}$$

$$G_{1,3}^{3,1}\left(z \,\middle|\, \begin{matrix} a+\frac{1}{2} \\ a+\frac{1}{2}, -a, a \end{matrix}\right) = \frac{\pi^2}{\cos 2\pi a}\, [\mathbf{H}_{2a}(2z^{1/2}) - Y_{2a}(2z^{1/2})]. \tag{37}$$

$$G_{1\,3}^{3\,1}\left(z\,\middle|\,\begin{matrix} a \\ a,\ b,\ -b \end{matrix}\right) = 2^{2-2a}\Gamma(1-a-b)\,\Gamma(1-a+b)S_{2a-1,\,2b}(2z^{1/2}) \tag{38}$$

$$G_{1\,3}^{3\,1}\left(z\,\middle|\,\begin{matrix} a+\frac{1}{2} \\ b,\ 2a-b,\ a \end{matrix}\right) = \frac{\pi^{5/2}z^{a}}{2\cos(b-a)\pi}\,H_{b-a}^{(1)}(z^{1/2})\,H_{b-a}^{(2)}(z^{1/2}) \tag{39}$$

See 6 4(20, 21) for G-functions with $m = q = 3$, $n = p = 1$

$$G_{2\,4}^{1\,2}\left(z\,\middle|\,\begin{matrix} a+\frac{1}{2},\ a \\ a+b,\ a-c,\ a+c,\ a-b \end{matrix}\right) = \pi^{1/2}z^{a}J_{b+c}(z^{1/2})\,J_{b-c}(z^{1/2}) \tag{40}$$

$$G_{2\,4}^{2\,2}\left(z\,\middle|\,\begin{matrix} a,\ a+\frac{1}{2} \\ b,\ c,\ 2a-c,\ 2a-b \end{matrix}\right) = 2\pi^{1/2}z^{a}I_{b+c-2a}(z^{1/2})\,K_{b-c}(z^{1/2}) \tag{41}$$

$$\begin{aligned} G_{2,4}^{3\,0}\left(z\,\middle|\,\begin{matrix} 0,\ \frac{1}{2} \\ a,\ b,\ -b,\ -a \end{matrix}\right) &= (i\pi^{1/2}/4)[H_{a-b}^{(1)}(z^{1/2})\,H_{a+b}^{(1)}(z^{1/2}) \\ &\quad - H_{a-b}^{(2)}(z^{1/2})\,H_{a+b}^{(2)}(z^{1/2})] \end{aligned} \tag{42}$$

$$\begin{aligned} G_{2\,4}^{3\,1}\left(z\,\middle|\,\begin{matrix} \frac{1}{4}+k,\ \frac{1}{4}-k \\ m-\frac{1}{4},\ \frac{1}{4},\ -\frac{1}{4},\ -m-\frac{1}{4} \end{matrix}\right) &= \frac{\pi^{1/2}\Gamma(m+\frac{1}{2}-k)}{z^{3/4}\Gamma(2m+1)} \\ &\quad \times M_{-k,\,m}(2z^{1/2})\,W_{k,\,m}(2z^{1/2}) \end{aligned} \tag{43}$$

$$G_{2\,4}^{4\,0}\left(z\,\middle|\,\begin{matrix} \frac{1}{2}+a,\ \frac{1}{2}-a \\ 0,\ \frac{1}{2},\ b,\ -b \end{matrix}\right) = (\pi/z)^{1/2}\,W_{a,\,b}(2z^{1/2})\,W_{-a,\,b}(2z^{1/2}) \tag{44}$$

$$G_{2\,4}^{4\,0}\left(z\,\middle|\,\begin{matrix} a,\ a+\frac{1}{2} \\ a+b,\ a+c,\ a-c,\ a-b \end{matrix}\right) = 2\pi^{-1/2}z^{a}K_{b+c}(z^{1/2})\,K_{b-c}(z^{1/2}) \tag{45}$$

$$\begin{aligned} G_{2\,4}^{4\,1}\left(z\,\middle|\,\begin{matrix} 0,\ \frac{1}{2} \\ a,\ b,\ -b,\ -a \end{matrix}\right) &= \frac{\pi^{5/2}}{4\sin\pi a\,\sin\pi b} \\ &\quad \times [e^{-i\pi b}H_{a-b}^{(1)}(z^{1/2})\,H_{a+b}^{(2)}(z^{1/2}) \\ &\quad - e^{i\pi b}H_{a+b}^{(1)}(z^{1/2})\,H_{a-b}^{(2)}(z^{1/2})] \end{aligned} \tag{46}$$

$$\begin{aligned} G_{2,4}^{4\,1}\left(z\,\middle|\,\begin{matrix} \frac{1}{2},\ 0 \\ a,\ b,\ -b,\ -a \end{matrix}\right) &= \frac{\pi^{5/2}}{4\cos\pi a\,\cos\pi b} \\ &\quad \times [e^{-i\pi b}H_{a-b}^{(1)}(z^{1/2})\,H_{a+b}^{(2)}(z^{1/2}) \\ &\quad + e^{i\pi b}H_{a+b}^{(1)}(z^{1/2})\,H_{a-b}^{(2)}(z^{1/2})] \end{aligned} \tag{47}$$

$$\begin{aligned} G_{2\,4}^{4\,1}\left(z\,\middle|\,\begin{matrix} \frac{1}{2}+a,\ \frac{1}{2}-a \\ 0,\ \frac{1}{2},\ b,\ -b \end{matrix}\right) &= (\pi/z)^{1/2}\,\Gamma(\tfrac{1}{2}+b-a)\,\Gamma(\tfrac{1}{2}-b-a) \\ &\quad \times W_{a,\,b}(2iz^{1/2})\,W_{a,\,b}(-2iz^{1/2}) \end{aligned} \tag{48}$$

Chapter VII ASYMPTOTIC EXPANSIONS OF ${}_pF_q$ FOR LARGE PARAMETERS

7.1. Introduction

The behavior of ${}_pF_q$ for large values of the variable has been discussed in 5.11. The behavior for large values of the parameters is much more complicated. No complete theory is available, and there are many open questions, especially as concerns uniform asymptotic expansions with respect to the variable z. Some results bearing on the subject of this chapter have already been given in 3.5 and 4.8.

7.2. The ${}_2F_1$

In the case of a large denominator parameter, we have

$$
\begin{aligned}
{}_2F_1(a, b; c; z) &= \sum_{k=0}^{m} \frac{(a)_k (b)_k z^k}{(c)_k k!} + R_{m+1}, \\
R_{m+1} &= O(c^{-m-1}), \qquad |\arg(1-z)| < \pi, \\
|c| \to \infty, &\qquad |\arg c| \leqslant \pi - \epsilon, \qquad \epsilon > 0,
\end{aligned} \tag{1}
$$

where a, b, and z are fixed. For the proof, using 6.2.1(7, 8), 3.6(10), and 3.6(1), we have

$$
\begin{aligned}
R_{m+1} = {} & \frac{\Gamma(c)\Gamma(a+m+1) z^{m+1}}{m!\, \Gamma(b)\Gamma(c-b)\Gamma(a)} \\
& \times \int_0^1 \int_0^1 t^{b+m}(1-t)^{c-b-1}(1-ztu)^{-a-m-1}(1-u)^m \, dt\, du,
\end{aligned}
$$

valid for $R(c-b) > 0$, $R(b+m) > -1$, and $|\arg(1-z)| < \pi$. We suppose m is sufficiently large so that the second inequality is true. If $0 \leqslant t \leqslant 1$, $0 \leqslant u \leqslant 1$, $|(1-ztu)^{-\alpha-m-1}| \leqslant M^{-\alpha-m-1}$, $\alpha = R(a)$ where M depends on z and is either the max or min of $|1 - ztu|$. Let $\beta = R(b)$ and $\gamma = R(c)$. Using 2.6(3) and 2.11(11), we find

$$
|R_{m+1}| \leqslant \left|\frac{z}{M}\right|^{m+1} M^{-\alpha} \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(m+1)!} \left|\frac{\Gamma(\beta)}{\Gamma(b)}\right| \left|\frac{\Gamma(c)}{\Gamma(c-b)}\right| \gamma^{-\beta-m-1}[1 + O(\gamma^{-1})],
$$

which proves (1) with $|R_{m+1}|$ instead of $O(c^{-m-1})$ provided m is sufficiently large But (1) is also valid for $m = 1, 2,$, since each term in the series on the right-hand side of (1) behaves like $|R_{m+1}|$ Another representation for the ${}_2F_1$ for large denominator parameter is given by (12)–(15) See also 4 8(8) with $h = 0$

The case when a single numerator parameter of a ${}_2F_1$ is large is of interest This, however, as well as the situation described by (1) are special cases of some results due to Watson (1918) who studied the ${}_2F_1$ where two or more of the parameters become large To present some of his findings, we first need some definitions Let

$$z = \cosh \xi, \qquad \xi = \mu + i\nu, \qquad \mu \geqslant 0 \qquad -\pi \leqslant \nu \leqslant \pi \tag{2}$$

These conditions determine ξ uniquely for a given z except when z is real, $z < 1$ It is supposed that the arguments of z, $z - 1$, and $z + 1$ are given their principal values (numerically not exceeding π), and in the special case when $z - 1$ is real and negative, it is supposed that z attains its value by a limiting process which then determines if $\arg(z - 1)$ is π or $-\pi$ The values of $\arg z$ and $\arg(z + 1)$ are determined in this special case in the same manner Thus,

$$e^{\pm\xi} = z \pm (z^2 - 1)^{1/2} \tag{3}$$

We also put

$$(1 - e^{\xi}) = (e^{\xi} - 1)e^{\mp i\pi} \tag{4}$$

where the upper (lower) sign is taken if $I(z) > (<) 0$ It is convenient to define two sectors in the complex λ-plane as follows

$$\text{Sector } P \qquad |\arg \lambda| \leqslant \pi - \delta, \qquad \delta > 0 \tag{5}$$

$$\begin{aligned} \text{Sector } Q \qquad & -\tfrac{1}{2}\pi - \omega_2 + \delta < \arg \lambda < \tfrac{1}{2}\pi + \omega_1 - \delta, \qquad \delta > 0, \\ & \omega_2 = \arctan(\nu/\mu), \qquad -\omega_1 = \arctan\{(\nu - \pi)/\mu\}, \qquad \nu \geqslant 0 \\ & \omega_2 = \arctan\{(\nu + \pi)/\mu\} \qquad -\omega_1 = \arctan(\nu/\mu) \qquad \nu < 0 \end{aligned} \tag{6}$$

where arc tan is given its principal value, that is, $|\arctan x| \leqslant \pi/2$, x real

Watson proves that

$$\begin{aligned} & {}_2F_1\left(\begin{matrix} a + \lambda, 1 + a - c + \lambda \\ 1 + a - b + 2\lambda \end{matrix} \middle| \frac{2}{1 - z}\right) \\ & \sim \frac{2^{a+b}\Gamma(1 + a - b + 2\lambda)(\pi/\lambda)^{1/2}}{\Gamma(1 + a - c + \lambda)\Gamma(c - b + \lambda)} \\ & \times \frac{[(z - 1)/2]^{a+\lambda} e^{(a-\lambda)\xi}(1 + e^{-\xi})^{c-a-b-1/2}}{(1 - e^{-\xi})^{c-1/2}} \sum_{k=0}^{\infty} f_k(\xi)(\tfrac{1}{2})_k \lambda^{-k}, \end{aligned} \tag{7}$$

valid when $|\lambda|$ is large over sector P [see (5)] and

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a+\lambda, b-\lambda \\ c \end{matrix}\middle|\frac{1-z}{2}\right) \sim{}& \frac{2^{a+b-1}\Gamma(1-b+\lambda)\Gamma(c)(1+e^{-\xi})^{c-a-b-1/2}}{(\lambda\pi)^{1/2}\Gamma(c-b+\lambda)(1-e^{-\xi})^{c-1/2}} \\
&\times\Big[e^{(\lambda-b)\xi}\sum_{k=0}^{\infty} f_k(-\xi)(\tfrac{1}{2})_k\lambda^{-k} \\
&+\exp[\pm i\pi(c-\tfrac{1}{2})]e^{-(\lambda+a)\xi}\sum_{k=0}^{\infty} f_k(\xi)(\tfrac{1}{2})_k\lambda^{-k}\Big], \qquad (8)
\end{aligned}
$$

valid when $|\lambda|$ is large over sector Q, see (6). In the latter formula, the upper (lower) sign is taken if $I(z) > (<)\, 0$. $f_k(\xi)$ is defined as follows:

$$
\begin{aligned}
&(1-e^{-2\xi})^k f_k(\xi) = \text{coefficient of } T^{2k} \text{ in the expansion of} \\
&\quad\left(1-\frac{T}{1-e^{-\xi}}\right)^{a-c}\left(1+\frac{T}{1+e^{-\xi}}\right)^{c-b-1} \\
&\qquad\times(1-T\operatorname{csch}\xi)^{-a}\left[\frac{1-e^{-2\xi}}{T^2}\ln\left(1+\frac{T^2}{1-(T+e^{-\xi})^2}\right)\right]^{-k-1/2} \qquad (9)
\end{aligned}
$$

in ascending powers of T, T sufficiently small. Clearly $f_0(\xi) = 1$ and

$$
\begin{aligned}
&f_1(\xi) = \tfrac{1}{2}(L + Me^{-\xi} + Ne^{-2\xi})/(1-e^{-2\xi}), \\
&\quad L = (a+b-2c+1)^2 - a + b - \tfrac{1}{2},\ \ M = -2(a+b-1)(a+b-2c+1), \\
&\quad N = (a+b-1)^2 + a - b + \tfrac{1}{2}. \qquad (10)
\end{aligned}
$$

A generalization of (8) is given in 7.4.2(8). See also 8.2(33–40).

For a future application, it is convenient to record

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a+\lambda, 1+a-c+\lambda \\ 1+a-b+2\lambda \end{matrix}\middle| z\right) \\
&\quad= \frac{2^{a+b}\Gamma(1+a-b+2\lambda)(\pi/\lambda)^{1/2}e^{-(a+\lambda)\varphi}(1-e^{-\varphi})^{c-a-b-1/2}[1+O(\lambda^{-1})]}{\Gamma(1+a-c+\lambda)\Gamma(c-b+\lambda)z^{a+\lambda}(1+e^{-\varphi})^{c-1/2}}, \\
&\qquad e^{\pm\varphi} = [2 - z \pm 2(1-z)^{1/2}]/z, \qquad (11)
\end{aligned}
$$

which is readily deduced from (7) if there we replace z by $(z-2)/z$.

Watson also proves that when λ is large,

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, b \\ c+\lambda \end{matrix}\middle| z\right) \sim \frac{\Gamma(c+\lambda)\lambda^{-b}}{\Gamma(c-b+\lambda)}\sum_{k=0}^{\infty} h_k(z)(b)_k\lambda^{-k}, \\
&\text{if}\quad |1-z^{-1}| \leqslant 1, \quad \lambda \text{ is in sector } P, \quad \text{see (5)}, \\
&\text{if}\quad |1-z^{-1}| > 1, \quad \lambda \text{ is in sector } Q, \quad \text{see (6)}, \qquad (12)
\end{aligned}
$$

and the coefficients $h_k(z)$ are defined by the relation

$$(1 - e^{-t})^{b-1} e^{-t(c-b)}(1 - z + ze^{-t})^{-a} = t^{b-1} \sum_{k=0}^{\infty} h_k(z) t^k, \qquad |t| < 1 \tag{13}$$

Clearly $h_0(z) = 1$ and by direct computation

$$h_1(z) = az + \tfrac{1}{2}(b + 1 - 2c) \tag{14}$$

Thus under the stated conditions,

$${}_2F_1\left(\begin{matrix} a & b \\ c + \lambda \end{matrix} \middle| z\right) = 1 + (abz/\lambda) + (abz/2\lambda^2)[(a + 1)(b + 1)z - 2c] + O(\lambda^{-3}) \tag{15}$$

In the reference cited, Watson gives a complete set of formulas to describe the asymptotic behavior of the ${}_2F_1$ in the form

$${}_2F_1\left(\begin{matrix} a + e_1\lambda, b + e_2\lambda \\ c + e_3\lambda \end{matrix} \middle| z\right) \tag{16}$$

where e_1, e_2, e_3 have the values 0, ± 1 The various cases are essentially of four types, as shown in the accompanying tabulation In the sequel, we consider only the type A and B situations

Case	e_1	e_2	e_3	Type
1	1	−1	0	A
2	1	1	0	A
3	−1	−1	0	A
4	0	0	1	B
5	0	0	−1	B
6	0	1	0	B
7	0	−1	0	B
8	0	1	1	B
9	0	−1	−1	B
10	1	1	1	B
11	−1	−1	−1	B
12	0	1	−1	C
13	0	−1	1	C
14	1	−1	1	C
15	1	−1	−1	C
16	1	1	−1	D
17	−1	−1	1	D

(17)

Case 1 is given by (8). If we employ Kummer's formulas 3.8 and the analytic continuation formula in 3.9, together with the expansions (7), (8), we can get asymptotic formulas for Cases 2 and 3 in (17). We now illustrate this idea which is also applicable for the cases under types B, C, and D. Consider Case 2 of (17). From 3.8(3),

$$ {}_2F_1\left(\begin{matrix} a+\lambda, b+\lambda \\ c \end{matrix}\middle| z\right) = (1-z)^{-a-\lambda}\, {}_2F_1\left(\begin{matrix} a+\lambda, c-b-\lambda \\ c \end{matrix}\middle| \frac{z}{z-1}\right). \tag{18} $$

Thus with an obvious change of notation, Case 2 stems from Case 1. For an alternative approach, using 3.9(3), and 3.8(1, 7, 17), we have

$$ \begin{aligned} &\frac{\Gamma(1+a-b)\Gamma(c-1)e^{i\pi(c-1)}}{\Gamma(a)\Gamma(c-b)} z^{1-c}(1-z)^{c-a-1}\, {}_2F_1\left(\begin{matrix} 1+a-c, 1-b \\ 2-c \end{matrix}\middle| \frac{z}{z-1}\right) \\ &\quad = (z^{-1}e^{i\pi})^a\, {}_2F_1\left(\begin{matrix} a, a+1-c \\ a+1-b \end{matrix}\middle| z^{-1}\right) - \frac{\Gamma(1+a-b)\Gamma(1-c)}{\Gamma(1-b)\Gamma(1+a-c)}\, {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| z\right). \end{aligned} \tag{19} $$

Now replace a, b, and c by $1+a-c+\lambda$, $1-b-\lambda$, and $2-c$, respectively. Then the ${}_2F_1$ on the left of (19) is of the form Case 2, while the first and second ${}_2F_1$'s on the right of (19) are of the form (7) and (8), respectively, when a slight change of notation is made. Again Case 3 may be obtained from Case 2 in view of 3.8(2). There are other possibilities. To enumerate these, it is convenient to use the notation $w_{i,j}$ defined in the introduction to the "Table of Solutions in the Degenerate Case" which appears in 3.10. Thus (19) is the relation which connects the solutions $w_{2,3}$, $w_{5,1}$, and $w_{1,1}$. In the sequel, we designate the function $w_{i,j}$ by the numerals ij for short, so that (19) connects the functions 23, 51, and 11. Similarly 3.8(3) connects 11 and 13. We form the following tabulation.

HYPERGEOMETRIC FUNCTIONS OF TYPE A

Coefficients of λ				
e_1	e_2	e_3	Functions	Case
1	−1	0	11, 12, 21, 22, 31, 32, 41, 42	1
1	1	0	13, 23, 33, 44	2
−1	−1	0	14, 24, 34, 43	3
1	1	2	51, 52, 53, 54	—
−1	−1	−2	61, 62, 63, 64	—

(20)

The first three columns give the coefficients of λ which define the functions to be investigated [see (16)]. These are listed in the fourth

column The simplest procedure is to use the continuation formulas of 3 9 or the formulas of 3 8 as appropriate to express a function of the fourth column in terms of the fundamental solutions 11 and 51, and then connect 11 and 51 with (7) and (8), respectively In our illustration of Case 2 we used functions 13 and 23, which are in the second row of the above table Alternatively, we could have used the relation connecting 33, 11, and 51 which follows from 3 9(13)

HYPERGEOMETRIC FUNCTIONS OF TYPE B

Coefficients of λ				
ϵ_1	ϵ_2	ϵ_3	Functions	Case
0	0	1	11 42	4
0	0	−1	22 31	5
0	1	0	52 53, 62 63	6
0	−1	0	51 54 61 64	7
0	1	1	13 14 43 44	8
0	−1	−1	23 24 33 34	9
1	1	1	12 41	10
−1	−1	−1	21 32	11

(21)

The use of this table is like that for type A In the present instance the fundamental solutions are 11 and 42 Since both are included in Case 4, it is sufficient to have the asymptotic expansion of only one of these functions The expansion for 11 is given by (12) To illustrate use of this table we develop the asymptotic expansion for ${}_3F_1(a, b + \lambda, c, z)$, i e, Case 6 Suppose we use function 52 As we need the relation between functions 52, 11, and 42, we employ the continuation formula 3 9(15) and use 3 8(1, 14, 18) In this relation, which is valid for $0 < \arg z < 2\pi$, replace a, b, and c by $c - a$, $1 - a$, and $1 - a + b + \lambda$, respectively If z is replaced by z^{-1}, and $e^{i\pi}$ is replaced by $e^{-i\pi}$, the resulting formula is also valid for $0 < \arg z < 2\pi$ We get

$$\frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)}\,{}_2F_1\left(\begin{matrix} a, b+\lambda \\ c \end{matrix}\middle| z\right) = \frac{\Gamma(1+b-c+\lambda)\Gamma(c-a)}{\Gamma(1+b-a+\lambda)} z^{a-c}(1-z)^{c-a-b-\lambda}$$
$$\times\, {}_2F_1\left(\begin{matrix} c-a, 1-a \\ 1+b-a+\lambda \end{matrix}\middle| z^{-1}\right)$$
$$+ \frac{\Gamma(1+b-c+\lambda)\Gamma(a)}{\Gamma(1+a+b-c+\lambda)} e^{i\pi a} z^{-a}$$
$$\times\, {}_2F_1\left(\begin{matrix} a, 1+a-c \\ 1+a+b-c+\lambda \end{matrix}\middle| 1-z^{-1}\right) \quad (22)$$

This could also be derived from the formula connecting the functions 11, 33, and 62. Now combine (22) with (12) and use 2.11(11) to get

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, b+\lambda \\ c \end{matrix}\middle| z\right) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\{\Gamma(c-a)(\lambda z)^{a-c}(1-z)^{c-a-b-\lambda}[1+O(\lambda^{-1})] \\
&\quad + \Gamma(a)(\lambda z e^{-i\pi})^{-a}[1+O(\lambda^{-1})]\}, \\
|\lambda| \to \infty, \quad \lambda \text{ in } Q \text{ if } |1-z| &\neq 1, \qquad \lambda \text{ in } P \text{ if } |1-z| = 1. \qquad (23)
\end{aligned}
$$

Note that (23) is exact when $a = c$ and $[1 + O(1/\lambda)]$ is replaced by unity. Similarly from the formula connecting the functions 11, 43, and 51, we get

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, b-\lambda \\ c \end{matrix}\middle| z\right) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\{\Gamma(a)(\lambda z)^{-a}[1+O(\lambda^{-1})] \\
&\quad + \Gamma(c-a)(\lambda z e^{-i\pi})^{a-c}(1-z)^{c-a-b+\lambda}[1+O(\lambda^{-1})]\}, \qquad (24)
\end{aligned}
$$

valid under the same conditions as for (23). Equation (24) may also be found from (22) in view of 3.8(3).

For a final example, we consider Case 5 of (21). Here it is convenient to consider the formula connecting 11, 22, and 31, see 3.10(5) and 3.9(1, 6, 9) with c replaced by $c - \lambda$. Thus,

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, b \\ c-\lambda \end{matrix}\middle| z\right) &= \frac{\Gamma(a+1-c+\lambda)\Gamma(b+1-c+\lambda)}{\Gamma(a+b+1-c+\lambda)\Gamma(1-c+\lambda)} \\
&\quad \times \left[{}_2F_1\left(\begin{matrix} a, b \\ a+b+1-c+\lambda \end{matrix}\middle| 1-z\right)\right. \\
&\quad - \frac{\pi\Gamma(a+b+1-c+\lambda)z^{1-c+\lambda}(1-z)^{c-a-b-\lambda}}{\Gamma(a)\Gamma(b)\Gamma(2-c+\lambda)\sin\pi(\lambda-c)} \\
&\quad \left. \times {}_2F_1\left(\begin{matrix} 1-a, 1-b \\ 2-c+\lambda \end{matrix}\middle| z\right)\right], \\
\lambda - c &\neq 0, \pm 1, \pm 2, \ldots, \qquad |\arg(1-z)| < \pi. \qquad (25)
\end{aligned}
$$

It is convenient to take λ large and positive. Using (12) and 2.11(11), we have

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, b \\ c-\lambda \end{matrix}\middle| z\right) &= 1 - \frac{abz}{\lambda} + O(\lambda^{-2}) - \frac{\pi\lambda^{a+b-1}[z/(1-z)]^{\lambda-c}z}{\Gamma(a)\Gamma(b)\sin\pi(\lambda-c)(1-z)^{a+b}} \\
&\quad \times \left\{1 + \frac{(a+b-1)(a+b+2-2c) - 2ab + 2(1-a)(1-b)z}{2\lambda} + O(\lambda^{-2})\right\}, \\
\lambda - c &\neq 0, 1, 2, \ldots, \qquad |\arg(1-z)| < \pi. \qquad (26)
\end{aligned}
$$

Now $|z/(1-z)| < 1$ if $R(z) < \frac{1}{2}$ so that

$$ {}_2F_1\left(\begin{matrix} a, b \\ c-\lambda \end{matrix}\middle| z\right) = 1 - (abz/\lambda) + O(\lambda^{-2}) \tag{27} $$

uniformly in any finite subregion of $R(z) < \frac{1}{2}$. If $z = \frac{1}{2}$, (26) shows that

$$ {}_2F_1\left(\begin{matrix} a, b \\ c-\lambda \end{matrix}\middle| \tfrac{1}{2}\right) = 1 - \frac{ab}{2\lambda} + O(\lambda^{-2}) - \frac{\pi(2\lambda)^{a+b-1}}{\Gamma(a)\Gamma(b)\sin\pi(\lambda-c)} $$
$$ \times \left\{1 + \frac{(a+b-1)(a+b+2-2c) - 2ab + (1-a)(1-b)}{2\lambda} + O(\lambda^{-2})\right\}, $$
$$ \lambda - c \neq 0, 1, 2, \tag{28} $$

Perron (1915–1918) also studied the ${}_2F_1$ for large parameters and has proved that

$$ {}_2F_1\left(\begin{matrix} a, b \\ c-r \end{matrix}\middle| z\right) \sim \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c-r)_n n!} + \frac{\pi\Gamma(a+b-c+r)}{\Gamma(a)\Gamma(b)\Gamma(1-c+r)\sin\pi(c-r)} $$
$$ \times z^{1-c+r}(1-z)^{c-a-b-r} \sum_{n=0}^{\infty} \frac{(1-a)_n(1-b)_n(1-z)^n}{(c+1-a-b-r)_n n!}, $$
$$ r \to +\infty, \quad r - c \neq 0, 1, 2, \ldots, \quad |\arg(1-z)| < \pi \tag{29} $$

This completes our discussion of functions of types A and B. The functions of types C and D do not seem to occur in the applications. So we dispense with further comments and merely refer the reader to Watson (1918). See also Perron (1915–1918).

7.3 Some Generalizations of the ${}_2F_1$ Formulas

Knottnerus (1960) has shown that for z fixed, r sufficiently large and positive,

$$ {}_{p+1}F_p\left(\begin{matrix} a_1+r, \ldots, a_{k-1}+r, a_k, \ldots, a_{p+1} \\ b_1+r, \ldots, b_k+r, b_{k+1}, \ldots, b_p \end{matrix}\middle| z\right) = \sum_{n=0}^{m} t_n + O(r^{-m-1}) $$
$$ 1 \leqslant k \leqslant p, \quad |\arg(1-z)| < \pi, \tag{1} $$

where t_n is the nth term in the ${}_{p+1}F_p$ series expansion. From 7.2(12) and 3.8(2), for r sufficiently large and positive,

$$ {}_2F_1\left(\begin{matrix} a+r, b+r \\ c+r \end{matrix}\middle| z\right) $$
$$ = (1-z)^{c-a-b-r}\left[1 + \frac{(c-a)(c-b)z}{r} + \sum_{k=2}^{n} \frac{c_k}{r^k} + O(r^{-n-1})\right], $$
$$ c_k = \sum_{s=1}^{k} \alpha_{k,s} z^s, \quad 2 \leqslant k \leqslant n, \quad |\arg(1-z)| < \pi, \tag{2} $$

where the $\alpha_{k,s}$ quantities depend only on a, b, and c, and more generally, Knottnerus shows that

$$ {}_{p+1}F_p\left(\begin{matrix} a_{p+1}+r \\ b_p+r \end{matrix}\middle| z\right) = (1-z)^\sigma\left[1+(d_1z/2r)+\sum_{k=2}^{n}(d_k/r^k)+O(r^{-n-1})\right], $$

$$ \sigma = \sum_{j=1}^{p} b_j - \sum_{j=1}^{p+1} a_j - r, \qquad d_1 = (\sigma+r)^2 - \sum_{j=1}^{p+1} a_j^2 + \sum_{j=1}^{p} b_j^2, $$

$$ d_k = \sum_{s=1}^{k} \beta_{k,s} z^s, \qquad 2 \leqslant k \leqslant n, \qquad |\arg(1-z)| < \pi. \tag{3} $$

Here the $\beta_{k,s}$ quantities depend only on the parameters a_h and b_h. We sketch the proof of (3) which is by induction. If $p = 1$, (3) and (2) are the same. Assume (3) is true for some integer $p \geqslant 1$. In (3) replace z by zt, multiply both sides by $t^{a+r-1}(1-t)^{b-a-1}$, and integrate with the aid of 3.6(10) and use (2). It readily follows that (3) holds for $p+1$ instead of p under the stated conditions, provided further that $R(b-a) > 0$. This last condition may be relaxed. For this and other details see the reference cited.

Another formula of interest is that for all z, z fixed, and r sufficiently large and positive,

$$ {}_pF_p\left(\begin{matrix} a_p+r \\ b_p+r \end{matrix}\middle| z\right) = e^z\left[1+(f_1z/r)+\sum_{k=2}^{n}(f_k/r^k)+O(r^{-n-1})\right], \qquad p > 0, $$

$$ f_1 = \sum_{j=1}^{p}(a_j - b_j), \qquad f_k = \sum_{s=1}^{k}\gamma_{k,s}z^s, \qquad 2 \leqslant k \leqslant n, \tag{4} $$

where $\gamma_{k,s}$ depends only on the parameters a_h and b_h. If $p = 1$, (4) follows from 4.4(12). Proof of (4) by induction is much like that for (3) where use is made of 4.2(1). Also for z fixed and r sufficiently large and positive,

$$ {}_pF_q\left(\begin{matrix} a_p+r \\ b_q+r \end{matrix}\middle| z\right) = 1+\sum_{k=0}^{n-2} t_k + O(r^{(p-q)n}), \qquad 0 \leqslant p < q, $$

$$ t_k = \frac{(a_p+r)_{k+1}z^{k+1}}{(b_q+r)_{k+1}(k+1)!}, \tag{5} $$

where $\sum_{k=0}^{n-2}$ is nil if $n = 1$. Both (4) and (5) have been given by Knottnerus (1960).

In (4) replace σ by z/t multiply both s des by $e^t t^{\beta-r}$ and integrate with the aid of 3 6(19) Then

$$
{}_pF_{p-1}\left(\begin{matrix} a_p + r \\ b_p + r \quad \beta + r \end{matrix}\middle| z\right) = {}_0F_1(\quad \beta + r \quad z)
$$

$$
+ \frac{f_1 z}{r(\beta + r)} {}_0F_1(\quad \beta + r + 1 \quad z) + O(r^{-2}) \qquad (6)
$$

valid for all σ z fixed r sufficiently large and positive This process may be iterated Note that each ${}_0F_1$ in (6) is related to a Bessel function see 4 4(6)

The inverse Laplace transform is an excellent technique to get further formulas Thus starting from (3) we can show that for all z σ fixed and r sufficiently large and positive

$$
{}_pF_{p-1}\left(\begin{matrix} a_{p+1} + r \\ 2r + \lambda + 1 \quad b_p + r \end{matrix}\middle| z\right) = e^{z/2}\left[1 + \frac{z^2 - 4z(\lambda + 1 - 2\theta)}{8(\lambda + 2r)} + O(r^{-2})\right] \qquad (7)
$$

$$
\theta = \sum_{j=1}^{p-1} a - \sum_{1}^{p} b
$$

For the proof use (3) with z replaced by z/t Multiply both sides by $e^t t^{\lambda-1-2r}$ and apply 3 6(19) with $c > |z|$ so that the restriction on $\arg(z/t)$ is fulfilled on the path of integrat on Then (7) readily follows from 4 8(16) In a similar fashion we have from (5) that for all z z fixed and r sufficiently large and positive

$$
{}_pF_{q-1}\left(\begin{matrix} a_p + r \\ 2r + \lambda + 1 \quad b_q + r \end{matrix}\middle| z\right) = 1 + \sum_{k=0}^{n-1} t_k^* + O(r^{p-q-1-n}) \qquad 0 \leqslant p \leqslant q
$$

$$
t_k^* = \frac{(a_p + r)_{k-1} z^{k-1}}{(b_q + r)_k \; (2r + \lambda + 1)_{k-1} (k + 1)!} \qquad (8)
$$

Next if r is suffic ently large and positive z fixed $z \neq 0$ $z \neq 1$ $|\arg(1 - z)| < \pi$ then

$$
{}_pF_{p+1}\left(\begin{matrix} a_{p-1} + r \quad \alpha + r \\ 2r + \lambda + 1 \quad b_p + r \end{matrix}\middle| z\right)
$$

$$
= \frac{2^{\lambda-2} \Gamma(2r + \lambda + 1)(\pi r)^{1/2} e^{-\alpha\phi}(1 + e^{-\phi})^{\beta-\alpha-1/2}[1 + O(r^{-1})]}{\Gamma(1 + \lambda - \theta + r)\Gamma(\theta + r) z^{-\alpha}(1 - e^{-\phi})^{\theta-\alpha-\lambda-1/2}}
$$

$$
e^{z\cosh\phi} \quad [2 - z + 2(1 - z)^{1/2}]/z \qquad \theta = \sum_{1}^{p-1} a - \sum_{1}^{p} b \qquad (9)
$$

To derive (9), use 3.6(10), (3), and 3.6(1). Thus,

$$\frac{\Gamma(\alpha+r)\Gamma(\lambda+1-\alpha+r)}{\Gamma(2r+\lambda+1)}\,{}_{p+2}F_{p+1}\left(\begin{matrix}a_{p+1}+r,\alpha+r\\2r+\lambda+1,b_p+r\end{matrix}\middle|z\right)$$

$$=\int_0^1 t^{\alpha-1+r}(1-t)^{\lambda-\alpha+r}\,{}_{p+1}F_p\left(\begin{matrix}a_{p+1}+r\\b_p+r\end{matrix}\middle|zt\right)dt$$

$$\sim\int_0^1 t^{\alpha-1+r}(1-t)^{\lambda-\alpha+r}(1-zt)^{-\theta-r}\left(1+\frac{d_1zt}{2r}+\frac{d_2}{r^2}+\cdots\right)dt,$$

$${}_{p+2}F_{p+1}\left(\begin{matrix}a_{p+1}+r,\alpha+r\\2r+\lambda+1,b_p+r\end{matrix}\middle|z\right)$$

$$\sim{}_2F_1\left(\begin{matrix}\alpha+r,\theta+r\\2r+\lambda+1\end{matrix}\middle|z\right)+\frac{d_1z(\alpha+r)}{2r(2r+\lambda+1)}\,{}_2F_1\left(\begin{matrix}\alpha+1+r,\theta+r\\2r+\lambda+2\end{matrix}\middle|z\right)$$

$$+\frac{z(\alpha+r)}{r^2(2r+\lambda+1)}\left\{\beta_{2,1}\,{}_2F_1\left(\begin{matrix}\alpha+1+r,\theta+r\\2r+\lambda+2\end{matrix}\middle|z\right)\right.$$

$$\left.+\frac{\beta_{2,2}z(\alpha+r+1)}{(2r+\lambda+2)}\,{}_2F_1\left(\begin{matrix}\alpha+2+r,\theta+r\\2r+\lambda+3\end{matrix}\middle|z\right)\right\}+\cdots.$$

Now employ 7.2(11) to get (9).

Generalization of the results 7.2(25–29) are also of interest. We prove that for z fixed and r sufficiently large and positive,

$${}_pF_{q+1}\left(\begin{matrix}a_p\\c-r,b_q\end{matrix}\middle|z\right)=\sum_{n=0}^{m}\frac{(a_p)_nz^n}{(c-r)_n(b_q)_nn!}+O(r^{-m-1}),$$

$$r-c\neq 0,1,2,\ldots;\quad p\leqslant q+1\quad\text{or}\quad p=q+2\quad\text{and}\quad R(z)<\tfrac{1}{2},\tag{10}$$

and

$${}_pF_{q+2}\left(\begin{matrix}a_p\\c-r,d+r,b_q\end{matrix}\middle|z\right)=\sum_{n=0}^{m}\frac{(a_p)_nz^n}{(c-r)_n(d+r)_n(b_q)_nn!}+O(r^{-2m-2}),$$

$$r-c\neq 0,1,2,\ldots;\quad p\leqslant q+2\text{ or }p=q+3\text{ and }|\arg(1-z)|<\pi.\tag{11}$$

For the proof, we start with 7.2(29). Replace z by zt, multiply by $t^{\alpha-1}(1-t)^{\beta-1}$ and use 3.6(1, 10) and 3.8(2). Then we can write

$${}_3F_2\left(\begin{matrix}a,b,\alpha\\c-r,\alpha+\beta\end{matrix}\middle|z\right)=U_m+V_m+W_m+X_m,$$

$$U_m=\sum_{n=0}^{m}\frac{(a)_n(b)_n(\alpha)_nz^n}{(c-r)_n(\alpha+\beta)_nn!},$$

$$V_m=\frac{\Gamma(\alpha+\beta)z^{m+1}}{r^{m+1}\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha+m-1}(1-t)^{\beta-1}f(m,zt)\,dt,$$

$$W_m = \frac{A\Gamma(\alpha+\beta)\Gamma(\alpha-c+r+1)}{\Gamma(\alpha)\Gamma(\alpha+\beta-c+r+1)}\left(\frac{z}{1-z}\right)^{1-c+r}$$

$$\times \sum_{n=0}^{m} h_n(1-z)^n\, {}_2F_1\left(\begin{matrix}\beta, \alpha+\beta+1-a-b+n\\ \alpha+\beta+1-c+r\end{matrix}\middle| z\right),$$

$$X_m = \frac{A\Gamma(\alpha+\beta)}{r^{m+1}\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \left(\frac{zt}{1-zt}\right)^{1-c+r}(1-t)^{\beta-1}(1-zt)^{m+2-a-b}g(m, zt)\,dt,$$

$$A = \frac{\pi\Gamma(a+b-c+r)}{\Gamma(a)\Gamma(b)\Gamma(1-c+r)\sin\pi(c-r)}, \quad h_n = \frac{(1-a)_n(1-b)_n}{(c+1-a-b-r)_n n!},$$

where $f(m, z)$ and $g(m, z)$ are bounded for all r uniformly in z, z fixed, $|\arg(1-z)| < \pi$ Clearly $V_m = O(r^{-m-1})$ If $R(z) < \frac{1}{2}$, $|z/(1-z)| < 1$ and W_m is subdominant to V_m In the integrand of X_m, replace t by $(1-u)/(1-uz)$ Then

$$X_m = \frac{A\Gamma(\alpha+\beta)(1-z)^{m+2+\beta-a-b}}{r^{m+1}\Gamma(\alpha)\Gamma(\beta)}\left(\frac{z}{1-z}\right)^{1-c+r}$$

$$\times \int_0^1 \frac{(1-u)^{1-c+r}u^{\beta-1}}{(1-uz)^{m+3+\beta-a-b}}\, g\left(m, \frac{z(1-u)}{1-uz}\right) du,$$

and so X_m is also subdominant to V_m This proves (10) for $p = 3$ and $q = 1$, provided $R(\alpha) > 0$, $R(\beta) > 0$ These conditions may be relaxed by appropriate use of loop integrals given in 3 6 Equation (10) follows by induction for general $p = q + 2$ by use of the beta transform Restrictions like $R(\alpha) > 0$, $R(\beta) > 0$ may be removed in another way, for consider

$${}_3F_2\left(\begin{matrix}a\ b, \alpha-m\\ c-r, \alpha+\beta-m\end{matrix}\middle| z\right) = \sum_{k=0}^{m-1}\frac{(a)_k(b)_k(\alpha-m)_k}{(c-r)_k(\alpha+\beta-m)_k}$$

$$+ \frac{(a)_m(b)_m(\alpha-m)_m}{(c-r)_m(\alpha+\beta-m)_m m!}$$

$$\times {}_4F_3\left(\begin{matrix}a+m, b+m\ \alpha, 1\\ c-r+m\ \alpha+\beta, 1+m\end{matrix}\middle| z\right)$$

where m is a positive integer such that $0 < R(\alpha) < m$, $0 < R(\alpha+\beta) < m$ Now (10) is valid for the above ${}_4F_3$ and so also for the ${}_3F_2$ For the case $p \leqslant q + 1$, (10) is readily proved using the inverse Laplace transform technique For example, in (10) replace z by z/t, multiply by $e^t t^{-s}$ and use 3 6(19) In the latter take $c > |2z|$ so that on the path of integration $R(z/t) < \frac{1}{2}$ Then (10) readily follows for $p = q + 1$ provided $R(\beta) > 0$ This can be relaxed so long as β is not a negative integer or zero by the

same argument used above to remove restrictions on the parameters incurred by use of the beta transform. Repeated use of 3.6(19) leads to the proof of (10) for $p < q + 1$.

Next we turn to the proof of (11). We proceed as in the proof of (10) but let $\beta = d - \alpha + r$. We then find that $V_m = O(r^{-2m-1})$. From 7.2(11), it follows that $W_m = O(e^{-r\omega})$, e^{ω} as in (9), $|e^{-\omega}| < 1$ for $|\arg(1 - z)| < \pi$. Thus W_m is subdominant to V_m. Since $g(m, zt)$ is bounded, it is straightforward to show, with the aid of 3.6(1) and 7.2(11), that $X_m = O(e^{-r\omega})$. This proves (11) for $p = 3$ and $q = 0$. Proof of (11) for general p and q as stated is very much like that for (10) and we dispense with further details.

7.4. Extended Jacobi Polynomials

7.4.1. Preliminary Results

Here we are concerned with the behavior of

$$F_n(z) = {}_{p+2}F_q\left(\begin{matrix}-n, n + \lambda, \alpha_p \\ \rho_q\end{matrix}\middle| z\right) \tag{1}$$

for large n. The subsequent developments follow closely the work of Fields and Luke (1963) and Fields (1965b).

If $p = 0$, $q = 1$, the asymptotic expansion for $F_n(z)$ for large n follows from 7.2(8). In this special case, $F_n(z)$ is essentially the Jacobi polynomial, see 8.2(46), if n is a positive integer. If $q = p + 1$, we call $F_n(z)$ a generalized Jacobi polynomial, and in general we call $F_n(z)$ the extended Jacobi polynomial.

In our work we need not insist that n be a positive integer unless $q < p + 1$, for otherwise (1) is a divergent series for all z, $z \neq 0$. Thus unless $q < p + 1$ or unless stated otherwise, we suppose that n is a large complex parameter. We also suppose that none of the ρ_h's, $h = 1, 2, \ldots, q$, is a negative integer or zero.

From 5.1(2),

$$[\delta(\delta + \rho_q - 1) + zN^{\beta}(\delta + \alpha_p) - z\delta(\delta + \lambda)(\delta + \alpha_p)]F_n(z) = 0,$$
$$N^{\beta} = n(n + \lambda), \qquad \beta = q + 1 - p, \tag{2}$$

which is of order $M = \max(p + 2, q + 1)$.

For our later work, we need to consider a function of the form

$$G(z) = A \exp\left\{N \int^{z} \tau(N, t)\, dt\right\}, \tag{3}$$

where A is a constant Where no confusion will result, we write

$$\tau(N, t) = \tau \tag{4}$$

τ is defined later by (6)

By induction on k, it can be shown that with $G = G(z)$,

$$\begin{aligned} G^{-1}\frac{d^kG}{dz^k} = (N\tau)^k &+ N^{k-1}\binom{k}{2}\tau^{k-2}\tau^{(1)} + N^{k-2}\left[\binom{k}{3}\tau^{k-3}\tau^{(2)} + 3\binom{k}{4}\tau^{k-4}(\tau^{(1)})^2\right] \\ &+ N^{k-3}\left[\binom{k}{4}\tau^{k-4}\tau^{(3)} + 10\binom{k}{5}\tau^{k-5}\tau^{(1)}\tau^{(2)} + 15\binom{k}{6}\tau^{k-6}(\tau^{(1)})^3\right] \\ &+ N^{k-4}\left[\binom{k}{5}\tau^{k-5}\tau^{(4)} + 5\binom{k}{6}\tau^{k-6}\{3\tau^{(1)}\tau^{(3)} + 2(\tau^{(2)})^2\}\right. \\ &\left. + 105\binom{k}{7}\tau^{k-7}(\tau^{(1)})^2\tau^{(2)} + 105\binom{k}{8}\tau^{k-8}(\tau^{(1)})^4\right] + H_{k-5}(N), \end{aligned} \tag{5}$$

where $\tau^{(m)}$ is the mth derivative of τ with respect to t, and $H_{k-5}(N)$ is a polynomial in N of degree $(k-5)$ whose coefficients depend on $\tau^{(j)}$, $j = 0, 1, \ldots, k$

Assume that

$$\tau(N, t) = \sum_{k=0}^{\infty} \frac{\tau_k(t)}{N^k} = \sum_{k=0}^{\infty} \frac{\tau_k}{N^k} \tag{6}$$

With the aid of 2 9(11–13), (2) can be expressed in terms of the differential operator D If we assume that $F_n(z)$ is a function of the form $G(z)$ as given by (3), and employ (5), formal substitution of these developments in (2) gives a power series in N equal to zero Then, if τ_k is chosen so that all coefficients of powers of N are zero, and $\tau_0 \neq 0$, a generally divergent series (6) is obtained With the help of this series and (3), formal solutions of (2) are obtained which serve as asymptotic representations of certain solutions of (2) in appropriate regions of the z-plane The equation for τ_0 is called the characteristic equation of the differential equation, and its behavior changes radically as $q = p + 1$ (Case 1), $q \leqslant p$ (Case 2), or $q \geqslant p + 2$ (Case 3)

7 4 2 Case 1, $q = p + 1$

Here $\beta = 2$, and we take

$$N^2 = n(n + \lambda) \tag{1}$$

Since infinity is a regular singular point of 7 4 1(2), $F_n(z)$ can be given as a linear combination of solutions around infinity of 7 4 1(2) Replacing

the solutions around infinity by their asymptotic representations for large n, we arrive at an asymptotic representation for $F_n(z)$, for large n.

A fundamental set of solutions for Case 1 of 7.4.1(2) contains $M = p + 2$ functions. There are p formal, algebraic, descending series solutions of 7.4.1(2) of the form

$$\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z) = \frac{(\alpha_p)^*_{-\alpha_t}}{(\rho_{p+1})_{-\alpha_t}}(z)^{-\alpha_t}$$

$$\times\, {}_{p+2}F_{p+1}\left(\begin{matrix}\alpha_t\,,\, 1+\alpha_t-\rho_{p+1}\\ 1+\alpha_t+n,\, 1+\alpha_t-n-\lambda,\, 1+\alpha_t-\alpha_p^*\end{matrix}\,\middle|\,\frac{1}{z}\right) \qquad (2)$$

where $t = 1, 2,\ldots, p$ and an asterisk means to delete the term when the subscripts t and p coincide. Observe that (2) and 5.11.1(7) differ only by a function independent of z if in the latter we replace p and q by $p + 2$ and $p + 1$, respectively, and set α_h equal to $-n$ and $n + \lambda$ for $h = p + 1$ and $p + 2$, respectively. Note that (2) and the functions in 5.1(27) are also closely related. For $|z| > 1$, the series defining $\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$ converge and are valid solutions of 7.4.1(2). Since n enters into two denominator parameters of the ${}_{p+2}F_{p+1}$ functions of (2), $\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$ may be considered as essentially an asymptotic expansion for large n of a valid solution of 7.4.1(2). Then for $0 < |z| \leqslant 1$, these same functions also serve as asymptotic expansions for large n to valid solutions of 7.4.1(2), see Erdélyi (1956) and the references given there. The $\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$'s then correspond to the p identically vanishing roots of the equation for τ_0, see 7.4.1(6). They are linearly independent if no two of the α_h's differ by an integer or zero. However, as previously noted, we can always get linearly independent solutions by taking limiting forms, see 5.1(27–30).

The lead terms of the exponential asymptotic expansions of the remaining $M - p = 2$ solutions of the fundamental set of solutions are computed by the formal procedure described above in the discussion surrounding 7.4.1(3–6), and are denoted by $\mathscr{K}^{(j)}_2(z)$, $j = 1, 2$. Here $\mathscr{K}^{(1)}_2(z)$ corresponds to the characteristic root $\tau_0 = i[z(1 - z)^{1/2}]$, and $\mathscr{K}^{(2)}_2(z)$ is $\mathscr{K}^{(1)}_2(z)$ with i replaced by $-i$. We may now write

$$F_n(z) \sim \sum_{t=1}^{p} A_t \mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z) + A_{p+1}\mathscr{K}^{(1)}_2(z) + A_{p+2}\mathscr{K}^{(2)}_2(z), \qquad (3)$$

where the A_t's, $t = 1, 2,\ldots, p + 2$, are connecting constants independent of z but dependent on n. We next determine the lead terms of the asymptotic expansions of the A_t's for $n \to \infty$. By the confluence

principle [see 3 5(34)], if $|z| < |n(n+\lambda)|$,

$$\lim_{n\to\infty} F_n\left(\frac{z}{n(n+\lambda)}\right) = \lim_{n\to\infty} {}_{p+2}F_{p+1}\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ \rho_{p+1}\end{matrix}\middle| \frac{z}{n(n+\lambda)}\right)$$

$$= {}_pF_{p+1}\left(\begin{matrix}\alpha_p \\ \rho_{p+1}\end{matrix}\middle| -z\right) \tag{4}$$

Thus, if in (3), z is replaced by $z/n(n+\lambda)$ while $0 \ll |z| \ll n(n+\lambda)$, then the A_t's can be found for large n by comparison with the asymptotic representation of ${}_pF_{p+1}(\alpha_p, \rho_{p+1}, -z)$ for large z, see 5 11 2(1) We find that

$$A_t \sim \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \sum_{j=0}^{\infty} c_j^{(t)} N^{-j}, \qquad c_0^{(t)} = 1, \quad t = 1, 2, \ldots, p, \tag{5}$$

$$A_{p+1} \sim \frac{N^{2\gamma}\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)\Gamma(\frac{1}{2})} e^{i\pi\gamma} \exp\left\{\sum_{j=1}^{\infty} [a_j + ib_j] N^{-j}\right\}, \tag{6}$$

and A_{p+2} is A_{p+1} with i replaced by $-i$ In 7 4 3 we will show by Darboux's method that

$$\begin{gathered} c_j^{(t)} = 0, \qquad j \geqslant 1, \quad t = 1, 2, \ldots, p, \\ b_1 = b_2 = b_3 = b_4 = 0, \qquad a_1 = a_3 = 0, \end{gathered} \tag{7}$$

and a_2 has the value given in (9) Thus,

$$\begin{aligned} {}_{p+2}F_{p+1}\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ \rho_{p+1}\end{matrix}\middle| z\right) \sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathcal{L}^{(\alpha_t)}_{p+2,p+1}(z) \\ + \frac{\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)\Gamma(\frac{1}{2})} N^{2\gamma}[\sin\theta/2]^{2\gamma}[\cos\theta/2]^{-2\gamma-\lambda} \\ \times \exp\{N^{-2}[\varphi_2(\theta) + a_2] + O(N^{-4})\} \\ \times \cos\{N\theta + \pi\gamma + N^{-1}\varphi_1(\theta) \\ + N^{-3}\varphi_3(\theta) + O(N^{-5})\}, \end{aligned}$$

$$|\arg z| \leqslant \pi - \epsilon, \qquad |\arg(1-z)| \leqslant \pi - \epsilon, \qquad \epsilon > 0, \tag{8}$$

where

$$\cos\theta = 1 - 2z \qquad \text{or} \qquad z = \sin^2\theta/2,$$

$$N^2 = n(n+\lambda), \qquad \gamma = (4)^{-1}(1 + 2B_1 - 2C_1),$$

$$B_1 = \sum_{t=1}^{p} \alpha_t, \qquad C_1 = \sum_{t=1}^{p+1} \rho_t,$$

$$B_2 = \sum_{s=2}^{p} \sum_{t=1}^{s-1} \alpha_s \alpha_t, \qquad C_2 = \sum_{s=2}^{p+1} \sum_{t=1}^{s-1} \rho_s \rho_t,$$

$$B_3 = \sum_{r=3}^{p} \sum_{s=2}^{r-1} \sum_{t=1}^{s-1} \alpha_r \alpha_s \alpha_t, \qquad C_3 = \sum_{r=3}^{p+1} \sum_{s=2}^{r-1} \sum_{t=1}^{s-1} \rho_r \rho_s \rho_t, \quad \text{etc.}$$

$$\varphi_1(\theta) = (\mu_1 + \mu_2)\cot(\theta/2) - 2(\mu_1 + \mu_2 + \mu_3)\cot\theta - \mu_1\theta/2,$$

$$\varphi_2(\theta) = \mu_4[\sec(\theta/2)]^2 + \mu_5[\csc(\theta/2)]^2,$$

$$\begin{aligned}\varphi_3(\theta) = {} & \tfrac{1}{3}[\mu_6 + \mu_7 + \mu_8 + \mu_9]\cot^3(\theta/2) + [\mu_8 + \mu_9 - \mu_6]\cot(\theta/2) \\ & - \tfrac{8}{3}[\mu_6 + \mu_7 + \mu_8 + \mu_9 + \mu_{10}]\cot^3\theta \\ & - 2[\mu_7 + 2\mu_8 + 3\mu_9 + 4\mu_{10}]\cot\theta + \mu_6\theta/2,\end{aligned}$$

$$\begin{aligned}a_2 = {} & -\lambda^2/16 + (2)^{-1}\lambda\{C_2 - B_2 + (2)^{-1}(B_1 - C_1)(B_1 + C_1 - 1)\} \\ & + (12)^{-1}\{3(B_1 + C_1) - 2(B_1^2 + B_1C_1 + C_1^2) - 1\}(B_1 - C_1) \\ & + (2)^{-1}\{B_2(B_1 - 1) + C_2(1 - C_1) + C_3 - B_3\},\end{aligned}$$

$$\mu_1 = -\lambda^2/4,$$

$$\mu_2 = (2)^{-1}(C_1 - B_1)(2B_1 + \lambda - 1) + B_2 - C_2 + 1/4,$$

$$\mu_3 = (4)^{-1}(B_1 - C_1)(3B_1 + C_1 - 2) + C_2 - B_2 - 3/16,$$

$$\mu_4 = (16)^{-1}(2\gamma + \lambda - 1)(2\gamma + \lambda) = -(4)^{-1}(\mu_1 + \mu_2 + \mu_3),$$

$$\begin{aligned}\mu_5 = {} & (16)^{-1}(C_1 - B_1)(8B_2 - 8B_1^2 + 11B_1 + C_1 - 2) \\ & + (4)^{-1}(C_2 - B_2)(2B_1 - 3) - (2)^{-1}(C_3 - B_3) + 3/64,\end{aligned}$$

$$\mu_6 = -\lambda^4/64,$$

$$\begin{aligned}\mu_7 = {} & (4)^{-1}(B_1 - C_1)[4B_3 + 10B_2 - 8B_1B_2 + 7B_1 + 4B_1^3 - 10B_1^2 \\ & + 2\lambda B_2 + 2\lambda B_1 - 2\lambda B_1^2 - \lambda^2 B_1/2 - \lambda^3/4 + \lambda^2/4 + \lambda/2 - 1/2] \\ & + (4)^{-1}(B_2 - C_2)[4B_2 - 4B_1^2 + 10B_1 + 2\lambda B_1 - 2\lambda + \lambda^2/2 - 7] \\ & + (2)^{-1}(2B_1 - \lambda - 5)(B_3 - C_3) + C_4 - B_4 + 3\lambda^2/32 - 1/16,\end{aligned}$$

$$\begin{aligned}\mu_{10} = {} & (64)^{-1}(B_1 - C_1)[C_1^3 + 5B_1C_1^2 + 35B_1^2C_1 - 105B_1^3 + 236B_1^2 \\ & + 160B_1B_2 - 24B_2C_1 - 8C_1C_2 - 40B_1C_1 - 4C_1^2 - 192B_2 - 64B_3 \\ & - 291B_1/2 - C_1/2 + 9] + (8)^{-1}(B_2 - C_2)[2C_2 - 10B_2 + 6C_1 \\ & - 30B_1 - 7B_1C_1 + 15B_1^2 + 73/4] \\ & + (2)^{-1}(B_3 - C_3)(C_1 - 3B_1 + 6) + B_4 - C_4 + 63/1024,\end{aligned}$$

$$
\begin{aligned}
2\mu_7 + \mu_8 - \mu_{10} = {} & (64)^{-1}(B_1 - C_1)[2B_1 - 8\lambda^2 B_1 - 8\lambda^3 + 4\lambda^2 + 24\lambda - 1] \\
& + (8)^{-1}(B_1 - C_1)^2[\lambda - 2\lambda B_1 - 3\lambda^2/4 + 23/16] \\
& + (64)^{-1}(B_1 - C_1)^3(4 - 7B_1 - C_1) \\
& + (8)^{-1}(B_2 - C_2)(B_1 - C_1)^2 \\
& + (B_2 - C_2)(B_1 - C_1)(\lambda/4) \\
& + (32)^{-1}(4\lambda^2 - 1)(B_2 - C_2) + 25\lambda^2/128 - 47/1024, \\
\mu_6 + \mu_7 + \mu_8 + \mu_9 + \mu_{10} = {} & -(64)^{-1}(2\gamma + \lambda - 3)(2\gamma + \lambda - 1)(2\gamma + \lambda)(2\gamma + \lambda + 2) \\
= {} & (2)^{-1}\mu_4(3 - 8\mu_4), \\
\mu_1 + \mu_2 + \mu_3 = {} & -(4)^{-1}(2\gamma + \lambda)(2\gamma + \lambda - 1), \\
\mu_6 + \mu_7 + \mu_8 + \mu_9 + \mu_{10} = {} & -(64)^{-1}(2\gamma + \lambda + 2)(2\gamma + \lambda) \\
& \times (2\gamma + \lambda - 1)(2\gamma + \lambda - 3), \\
\mu_7 + 2\mu_8 + 3\mu_9 + 4\mu_{10} = {} & 3(\mu_6 + \mu_7 + \mu_8 + \mu_9 + \mu_{10}) \\
& - (2\mu_7 + \mu_8 - \mu_{10}) - 3\mu_6
\end{aligned} \tag{9}
$$

The coefficient a_2 as given by Fields (1965b) deletes the factor $(B_1 - C_1)$ from the $(12)^{-1}\{\ \}$ term

The restrictions on arg z in (8) arise since the real axis is the Stokes line for ${}_pF_{p+1}(\alpha_p, \rho_{p+1}, -z)$, and the confluence on $F_n(z)$ effectively moves the singularity at unity of $F_n(z)$ out to infinity along the positive real axis

By induction, it can be shown that with the exception of $\pi\gamma$, only odd powers of N appear inside the cosine term of (8) with curly brackets, while only even powers of N appear in the exponential term

Representations valid along the negative real axis and positive real axis to the right of unity can be similarly constructed The connecting constants in this instance are determined from (8) by comparing the dominant terms in those overlapping regions where both representations are valid Hence

$$
\begin{aligned}
& {}_{p+2}F_{p+1}\left(\begin{matrix} -n, n + \lambda, \alpha_p \\ \rho_{p+1} \end{matrix} \middle| -z\right) \\
& \sim \sum_{i=1}^{p} \frac{(n + \lambda)_{-\alpha_i}}{(n + 1)_{\alpha_i}} \mathscr{L}^{(\alpha_i)}_{p+2,\, p+1}(ze^{i\delta\pi}) \\
& + \frac{\Gamma(\rho_{p+1})N^{2\gamma}}{\Gamma(\alpha_p)\Gamma(\frac{1}{2})}[\sinh(\xi/2)]^{-\gamma}[\cosh(\xi/2)]^{-2\gamma-\lambda} \\
& \times \exp\{N^{-2}[\varphi_2(\xi t) + a_2] + O(N^{-4})\} \\
& \times \cosh\{N\xi - iN^{-1}\varphi_1(\xi t) - iN^{-3}\varphi_3(\xi t) + O(N^{-5})\}, \\
& \cosh \xi = 1 + 2z, \qquad |\arg z| \leqslant \pi - \epsilon, \qquad \epsilon > 0, \\
& \delta = +(-) \quad \text{if} \quad \arg z < (>) 0
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
{}_{p+2}F_{p+1}\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ \rho_{p+1}\end{matrix}\middle| z\right) \sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z) \\
+ (-)^n \frac{\Gamma(\rho_{p+1})N^{2\nu}}{\Gamma(\alpha_p)\Gamma(\frac{1}{2})} [\cosh(\eta/2)]^{2\nu}[\sinh(\eta/2)]^{-2\nu-\lambda} \\
\times \exp\{N^{-2}(\varphi_2(\pi+\eta i) + a_2] + O(N^{-4})\} \\
\times \cosh\{N\eta - iN^{-1}[\varphi_1(\pi+\eta i) + \mu_1\pi/2] \\
- iN^{-3}[\varphi_3(\pi+\eta i) - \mu_6\pi/2] + O(N^{-5})\}, \\
\cosh\eta = 2z - 1, \qquad |\arg(z-1)| \leqslant \pi - \epsilon, \qquad \epsilon > 0.
\end{aligned} \tag{11}
$$

Note that in the development of (8), we could only infer the value of a_2 upon appeal to Darboux's method (see 7.4.3) which postulates that $F_n(z)$ is a polynomial. However, in some recent work Fields (1968) has shown that the results of the Darboux analysis remain valid for n not an integer. It should also be pointed out that the expansions (8)–(11) are not uniform in z. The work of Fields mentioned above contains uniform expansions of $F_n(z)$ for $0 \leqslant z \leqslant 1$. In particular, it is shown that the polynomials have a uniform algebraic rate of growth when $0 \leqslant z \leqslant 1$. Also it turns out that (8) is valid for $0 < z < 1$ and λ bounded so long as $N \sin\theta \to \infty$, and the first and second O-symbols in (8) are replaced by $O((N\sin\theta)^{-4})$ and $O((N\sin\theta)^{-5})$, respectively.

The representations (8)–(11) are essentially generalizations of 7.2(8) and agree in the $p = 0$ case up to terms of $O(n^{-1})$. The difference comes in the choice of the large parameter. Classically one puts $N = n + \lambda/2$, but from the differential equation 7.4.1(2), it is much more natural to set $N^2 = n(n+\lambda)$.

If λ is bounded, the expansions (8)–(11) can be improved as follows. Since

$$
\begin{aligned}
(n+\lambda/2) &= N(1+\lambda^2/4N^2)^{1/2}, \\
N &= n + \lambda/2 - N^{-1}(\lambda^2/8) + N^{-3}(\lambda^4/128) + O(N^{-5}),
\end{aligned} \tag{12}
$$

the cos or cosh terms with curly brackets in (8), (10), and (11) can be rewritten, respectively, as follows:

$$\cos\{(n+\lambda/2)\theta + \pi\gamma + N^{-1}\Phi_1(\theta) + N^{-3}\Phi_3(\theta) + O(N^{-5})\}, \tag{13}$$

$$\cosh\{(n+\lambda/2)\xi - iN^{-1}\Phi_1(\xi i) - iN^{-3}\Phi_3(\xi i) + O(N^{-5})\}, \tag{14}$$

$$\cosh\{(n+\lambda/2)\eta - iN^{-1}\Phi_1(\pi+\eta i) - iN^{-3}\Phi_3(\pi+\eta i) + O(N^{-5})\}, \tag{15}$$

$$\Phi_1(\theta) = \varphi_1(\theta) - \lambda^2\theta/8, \qquad \Phi_3(\theta) = \varphi_3(\theta) + \lambda^4\theta/128. \tag{16}$$

The choice $N^2 = n(n+\lambda)$ is advantageous since N large leads to the dual interpretation that either n or λ or both n and λ are large. Suppose now

that λ is not bounded We have found that if in our asymptotic developments (8)–(10), z is replaced by $z/(n+\lambda)$, $0 \leqslant n \leqslant \lambda$, then by confluence on λ (that is, let $\lambda \to \infty$), we can get like representations for ${}_{p+1}F_p(-n, \alpha_p, \rho_q, z)$ which include known results for the classical Laguerre polynomials However, further discussion is deferred to 7 4 6

Except for those values of z explicitly excluded, and the singular points zero, one, and infinity, (8), (10), (11) hold for all fixed z values as $N \to \infty$ For N fixed and z varying, we require that the correction terms in the above representations be small It appears sufficient to have the pragmatic restriction $N^2 z \to \infty$, $N^2(1-z) \to \infty$, and $\ln|z| \leqslant O(N)$ for z near zero, unity, and infinity, respectively

The case $0 < z \leqslant 1$ has been studied by Fields (1965b) in a different manner using Darboux's method of generating functions, and we now turn to 7 4 3 for this development

7 4 3 An Alternative Method for $q = p + 1$

In this section we use Darboux's classical method of generating functions [see, for example, Courant and Hilbert (1955) and Szego (1959)] to deduce the lead terms of the asymptotic expansion of 7 4 1(1) for large n when $0 < z \leqslant 1$ We suppose that n is a positive integer [but see the comments following 7 4 2 (11)] and consider

$$G_{p,q}(t, z) = (1-t)^{-\lambda}\, {}_{p+2}F_q\left(\begin{matrix}\alpha_p, \lambda/2, (\lambda+1)/2 \\ \rho_q\end{matrix}\middle| -\frac{4tz}{(1-t)^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!}\, {}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ \rho_q\end{matrix}\middle| z\right) \tag{1}$$

which is proved in 9 1(36) If z is fixed and $p \leqslant q+1$, (1) is valid near $t = 0$ We temporarily assume that

$$\alpha_j \quad \text{is not a negative integer or zero} \quad j = 1, 2, \ldots, p, \tag{2}$$

$$\lambda \quad \text{is not a negative integer or zero} \tag{3}$$

$$\alpha_i - \alpha_j \quad \text{is not an integer or zero} \quad i \neq j, \quad j = 1, 2, \ldots, p \tag{4}$$

We take $0 < z \leqslant 1$, $q = p + 1$, and write $G(t, z)$ for $G_{p,p+1}(z)$ Then $G(t, z)$ as a power series in t has radius of convergence unity and singularities at $t = 1$ and $t = e^{\pm i\theta}$ where

$$z = (1 - \cos\theta)/2 = \sin^2\theta/2 \tag{5}$$

In the Darboux analysis, we must know the behavior of $G(t, z)$ near these singularities

The behavior of $G(t, z)$ for t near unity $[-4tz/(1-t)^2$ near infinity$]$ follows from 5.3(3). Thus with $\alpha_h = \lambda/2$ and $\alpha_{h+1} = (\lambda+1)/2$ for $h = p+1$,

$$\begin{aligned} G(t, z) = &\sum_{h=1}^{p} \frac{(\alpha_{p+2})^*_{-\alpha_h}(1-t)^{2\alpha_h-\lambda}}{(\rho_{p+1})_{-\alpha_h}(4tz)^{\alpha_h}} {}_{p+2}F_{p+1}\left(\begin{matrix}1+\alpha_h-\rho_{p+1}, \alpha_h \\ 1+\alpha_h-\alpha^*_{p+2}\end{matrix}\middle| \frac{1}{w}\right) \\ &+ \frac{(\alpha_p)_{-\lambda/2}}{(\rho_{p+1})_{-\lambda/2}(\frac{1}{2})_{\lambda/2}(4tz)^{\lambda/2}} {}_{p+2}F_{p+1}\left(\begin{matrix}1+\lambda/2-\rho_{p+1}, \lambda/2 \\ 1+\lambda/2-\alpha_p, 1/2\end{matrix}\middle| \frac{1}{w}\right) \\ &+ \frac{(\alpha_p)_{-(\lambda+1)/2}\Gamma(-\frac{1}{2})(1-t)}{(\rho_{p+1})_{-(\lambda+1)/2}\Gamma(\lambda/2)(4tz)^{(\lambda+1)/2}} \\ &\times {}_{p+2}F_{p+1}\left(\begin{matrix}1+(\lambda+1)/2-\rho_{p+1}, (\lambda+1)/2 \\ 1+(\lambda+1)/2-\alpha_p, 3/2\end{matrix}\middle| \frac{1}{w}\right), \\ &w = -4tz(1-t)^{-2}, \qquad |\arg(-w)| < \pi. \end{aligned} \tag{6}$$

The behavior of $G(t, z)$ for t near $e^{\pm i\theta}$ is more complicated, but can be deduced from the work of Nörlund (1955). Since

$$(1-t)^2 + 4zt = (t - e^{-i\theta})(t - e^{i\theta}), \tag{7}$$

t near $e^{i\theta}$ implies that $w = -4zt(1-t)^{-2}$ is near unity. Nörlund's analysis shows that under conditions (2), (3)

$$\begin{aligned} &\frac{\Gamma(\frac{1}{2})\Gamma(\lambda)\Gamma(\alpha_p)}{2^{\lambda-1}\Gamma(\rho_{p+1})} {}_{p+2}F_{p+1}\left(\begin{matrix}\lambda/2, (\lambda+1)/2, \alpha_p \\ \rho_{p+1}\end{matrix}\middle| w\right) \\ &\quad = \Gamma(-\Delta_{p+2})(1-w)^{\Delta_{p+2}}\psi_{p+2}(w) + \varphi_{p+2}(w), \\ &\qquad\qquad \Delta_{p+2} \neq \text{integer or zero}; \\ &\quad = \frac{(-)^{1+\Delta_{p+2}}}{\Gamma(1+\Delta_{p+2})}(1-w)^{\Delta_{p+2}} \ln(1-w)\psi_{p+2}(w) + \varphi_{p+2}(w), \\ &\qquad\qquad \Delta_{p+2} = \text{a positive integer or zero}; \\ &\quad = \Gamma(-\Delta_{p+2})(1-w)^{\Delta_{p+2}}\chi_{p+2}(w) + (-)^{1+\Delta_{p+2}} \ln(1-w)\eta_{p+2}(w) + \varphi_{p+2}(w), \\ &\qquad\qquad \Delta_{p+2} = \text{a negative integer}; \end{aligned} \tag{8}$$

where

$$w = -4tz(1-t)^{-2}, \qquad \Delta_{p+2} = C_1 - B_1 - \lambda - \tfrac{1}{2} = -2\gamma - \lambda, \tag{9}$$

γ as in 7.4.2(9), and where $\psi_{p+2}(w)$, $\eta_{p+2}(w)$, and $\varphi_{p+2}(w)$ are analytic functions of w at $w =$ unity, and $\chi_{p+2}(w)$ is a polynomial in w of degree $-(1+\Delta_{p+2})$, $\psi_{p+2}(w)$ and $\chi_{p+2}(w)$ are normalized so that $\psi_{p+2}(1) = \chi_{p+2}(1) = 1$. For $\theta = \pi$, that is, $z = 1$, the singularities at $e^{\pm i\theta}$ coalesce. For this reason the analysis breaks down into the cases $0 < z < 1$ and $z = 1$. We consider the case $0 < z < 1$ first.

By Darboux's method, the coefficients of t^n in those terms of (6) and (8) which are singular at $t =$ unity and $w =$ unity, respectively, must be asymptotically equivalent to the coefficient of t^n in the right-hand side of (1) This leads to the representation 7 4 2(3) and to the evaluation of the coefficients A_h, $h = 1, 2, \ldots, p + 2$ [see 7 4 2(5–7)] We next illustrate the computation relating to (6)

Let $H(t)$ stand for the first p terms in the expansion (6) for $G(t, z)$ Under the assumption that $2\alpha_h - \lambda$ is not a positive integer or zero, $H(t)$ is not analytic at $t =$ unity Put

$$\Omega_h(t) = t^{-\alpha_h}\, {}_{p+2}F_{p+1}\left(\begin{matrix} 1 + \alpha_h - \rho_{p+1}, \alpha_h \\ 1 + \alpha_h - \alpha^*_{p+2} \end{matrix} \middle| -\frac{(1-t)^2}{4tz}\right), \tag{10}$$

and in place of t, write $1 - (1 - t)$ Then

$$\Omega_h(t) = \sum_{k=0}^{\infty} (-)^k \Omega_{h\,k} (1-t)^k, \qquad \Omega_{h\,0} = 1, \tag{11}$$

and

$$\frac{n!}{(\lambda)_n} H(t) = \frac{n!}{(\lambda)_n} \sum_{h=1}^{p} \frac{(\alpha_p)^*_{-\alpha_h} (\lambda/2)_{-\alpha_h} [(\lambda+1)/2]_{-\alpha_h}}{(\rho_{p+1})_{-\alpha_h} (4z)^{\alpha_h}} \sum_{k=0}^{\infty} (-)^k \Omega_{h\,k} (1-t)^{2\alpha_h - \lambda + k}$$

$$= \frac{n!}{(\lambda)_n} \sum_{h=1}^{p} \frac{(\alpha_p)^*_{-\alpha_h} (\lambda)_{-2\alpha_h}}{(\rho_{p+1})_{-\alpha_h} z^{\alpha_h}} \sum_{k=0}^{\infty} (-)^k \Omega_{h\,k} \sum_{s=0}^{\infty} \frac{(\lambda - 2\alpha_h - k)_s t^s}{s!}$$

$$= \frac{n!}{(\lambda)_n} \sum_{h=1}^{p} \frac{(\alpha_p)^*_{-\alpha_h} z^{-\alpha_h}}{(\rho_{p+1})_{-\alpha_h}} \sum_{k=0}^{\infty} \Omega_{h\,k} (1 - \lambda + 2\alpha_h)_k \sum_{s=0}^{\infty} \frac{(\lambda + s)_{2\alpha_h - k} (\lambda)_s t^s}{s!} \tag{12}$$

The coefficient of t^n in (12) is easily read and comparing with 7 4 2(3, 5), we see that

$$\frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathscr{L}^{(\alpha_t)}_{p+2,\,p+1}(z) \sum_{j=0}^{\infty} c_j^{(t)} N^{-j} \sim \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \sum_{j=0}^{\infty} d_j^{(t)} N^{-j}, \tag{13}$$

where $d_j^{(t)}$ is a rational expression in the α_j's whose coefficients depend on z The coefficients $c_j^{(t)}$ and $d_j^{(t)}$ are related as follows If $\mathscr{L}^{(\alpha_t)}_{p+2,\,p+1}(z)$ is expanded in powers of N^{-j} and this series is multiplied by $\sum_{j=0}^{\infty} c_j^{(t)} N^{-j}$, the resulting coefficients of N^{-j} must be $d_j^{(t)}$ since the Poincare coefficients are unique Suppose now that among the α_j's, one of these, call

it α_t, is a negative integer. It follows from 7.4.1(1), 3.2(3), and 7.4.2(2) that

$$F_n(z) = \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z), \tag{14}$$

and as (14) must be the same as the left-hand side of (13), it follows that $c_j^{(t)} = 0, j \geqslant 1$. But for j fixed, $c_j^{(t)}$ has only a finite number of distinct zeros as a rational expression in α_j. Also $c_j^{(t)}$ must vanish when α_j is any one of an infinity of negative integers. Hence $c_j^{(t)} \equiv 0, j \geqslant 1$. Allowing t to take on the values $1, 2, \ldots, p$, we get the first part of the statement 7.4.2(7).

The same kind of computation relating to (8) is straightforward. For $0 < z < 1$, the analysis leads to the representation of A_{p+1} as given by 7.4.2(6) and after some lengthy algebra the first few coefficients in the series for A_{p+1} are as stated in 7.4.2(7).

We now discuss the conditions (2)–(4). We can remove (2) in virtue of the remarks surrounding (14). Concerning (3), we can use the duplication formula for gamma functions to write

$$\begin{aligned}\Gamma(\lambda)G_{p,q}(t, z) &= (1-t)^{-\lambda} \sum_{k=0}^{\infty} \frac{\Gamma(\lambda+2k)(\alpha_p)_k}{(\rho_q)_k k!} \left[\frac{-zt}{(1-t)^2}\right]^k \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+n)t^n}{n!} {}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ \rho_q\end{matrix} \middle| z\right).\end{aligned} \tag{15}$$

Then, if λ approaches the nonpositive integer $(-m)$, only the first $(m+1)$ terms on the right of (15) would fail to be defined. But these $(m+1)$ terms can be removed by $(m+1)$ differentiations with respect to t. The analysis of $\partial^{m+1}\{\Gamma(\lambda)G(t, z)\}/\partial t^{m+1}$ can be carried out as before. Finally, if two or more of the α_j's differ by an integer or zero, then the representation (6) for $G(t, z)$ contains terms involving $\ln w$, see the remarks surrounding 5.1(27–30). But it can be shown that this would not invalidate the analysis based on Darboux's method. Thus the conditions (2)–(4) may be omitted.

We now turn to the situation when $z = 1$, $w = -4t/(1-t)^2$. In the application of Darboux's method, the analysis based on (6) when $0 < z < 1$ continues to apply when $z = 1$. As previously remarked, the singularities of $G(t, z)$, $t = e^{\pm i\theta}$, coalesce when $z = 1$, that is, $\theta = \pi$ and $w = 1$. From (1) and (8) we need $n!/(\lambda)_n$ times the coefficient of t^n in

$$G^* = \frac{(1-t)^{-\lambda}2^{\lambda-1}\Gamma(\rho_{p+1})}{\Gamma(\frac{1}{2})\Gamma(\lambda)\Gamma(\alpha_p)}\Gamma(\lambda+2\gamma)(1-w)^{-\lambda-2\gamma}\psi_{p+2}(w). \tag{16}$$

It is convenient to let

$$(1-t)^{\lambda+4\gamma}\psi_{p+2}(w) = 2^{\lambda+4\gamma}\sum_{k=0}^{\infty} g_k(1+t)^k, \qquad g_0 = 1 \tag{17}$$

By direct computation, we find from Norlund's analysis

$$\begin{aligned}
g_1 &= -[(\lambda+1)/2 + B_1 - C_1],\\
g_2 &= \frac{(\lambda - \frac{1}{2} + B_1 - C_1)^{-1}}{4}\Big[2\Big(\frac{\lambda+1}{2} + B_1 - C_1\Big)\Big(\frac{\lambda}{2} + B_1 - C_1\Big)\\
&\quad \times (\lambda - 1 + B_1 - C_1) + B_1(B_1 - C_1) - (B_2 - C_2)\Big],\\
g_3 &= -\frac{[(\lambda-1)/2 + B_1 - C_1]}{4(\lambda - \frac{1}{2} + B_1 - C_1)}\Big[\frac{2}{3}\Big(\frac{\lambda+1}{2} + B_1 - C_1\Big)\Big(\frac{\lambda}{2} + B_1 - C_1\Big)\\
&\quad \times (\lambda - 2 + B_1 - C_1) + B_1(B_1 - C_1) - (B_2 - C_2)\Big]
\end{aligned} \tag{18}$$

Combining (16) and (17), we have

$$\begin{aligned}
G^* &= \frac{2^{2\lambda+4\gamma-1}\Gamma(\lambda+2\gamma)\Gamma(\rho_{p+1})}{\Gamma(\frac{1}{2})\Gamma(\lambda)\Gamma(\alpha_p)}\sum_{k=0}^{\infty} g_k(1+t)^{k-4\gamma-2\lambda}\\
&= \frac{2^{2\lambda+4\gamma-1}\Gamma(\lambda+2\gamma)\Gamma(\rho_{p+1})}{\Gamma(\frac{1}{2})\Gamma(\lambda)\Gamma(\alpha_p)}\sum_{k=0}^{\infty} g_k \sum_{r=0}^{\infty}\frac{(-)^r(-k+4\gamma+2\lambda)_r t^r}{r!},
\end{aligned} \tag{19}$$

and with V_n equal to $n!/(\lambda)_n$ times the coefficient of t^n in (19), we get

$$V_n = \frac{(-)^n(\lambda+n)_{\lambda+4\gamma}\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)\Gamma(\lambda+2\gamma+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{g_k(1-4\gamma-2\lambda)_k}{(1-4\gamma-2\lambda-n)_k} \tag{20}$$

Now from 2 11(12) and the definition of N, for n sufficiently large,

$$(\lambda+n)_{\lambda+4\gamma} \sim n^{\lambda+4\gamma}\sum_{k=0}^{\infty} h_k n^{-k}, \qquad (\lambda+n)_{\lambda+4\gamma} \sim N^{\lambda+4\gamma}\sum_{k=0}^{\infty} h_k^* N^{-k}, \tag{21}$$

where the h_k's and the h_k^*'s are readily evaluated We can also develop for n sufficiently large

$$[(1-4\gamma-2\lambda-n)_k]^{-1} = N^{-k}\sum_{k=0}^{\infty} u_k N^{-k}, \tag{22}$$

and so we can derive an asymptotic expansion for V_n in descending powers of N

Putting all our results together for $z = 1$, we get

$$
{}_{p+2}F_{p+1}\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ \rho_{p+1}\end{matrix}\middle| 1\right) \sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathscr{L}^{(\alpha_t)}_{p+2,p+1}(1)
$$

$$
+ \frac{(-)^n \Gamma(\rho_{p+1}) N^{\lambda+4\gamma}}{\Gamma(\lambda+2\gamma+\frac{1}{2})\Gamma(\alpha_p)} \{1 + EN^{-2} + O(N^{-4})\}, \tag{23}
$$

where

$$
4\gamma = 1 + 2B_1 - 2C_1,
$$

$$
E = (24)^{-1}\{(\lambda + 1 + 2B_1 - 2C_1)[3\lambda^2 - (\lambda + 2B_1 - 2C_1)(\lambda + 2 + 2B_1 - 2C_1)]
$$

$$
+ 24(\lambda + B_1 - C_1)[B_1^2 - B_1C_1 + C_2 - B_2]\}, \tag{24}
$$

and the notation is as in 7.4.2(9). Had (20) been used as is, and $[(1 - 4\gamma - 2\lambda - n)_k]^{-1}$ developed as a descending series in $(n + \lambda)$, the last term of (23) would have contained nonzero terms of order $(n + \lambda)^{-r}$, $r = 1, 3$. The advantage in expanding in powers of $1/N$ is manifest.

A special case of (23) is interesting. Suppose $p = 1$ and the ${}_3F_2$ is Saalschützian, see 3.13.3(2, 4). This implies that $(\lambda + 2\gamma + \frac{1}{2})$ is nil and so the last term in (23) is nil.

S. O. Rice (1940) used Darboux's method for the polynomials ${}_3F_2(-n, n + 1, \zeta; 1, p; v)$ and obtained the analogs of 7.4.2(8) and (23) as modified by the discussion following (24). His analysis was simplified, however, by the fact that the generating function $G(t, z)$ of these polynomials reduces to the classical Gaussian hypergeometric function by cancellation of factors.

7.4.4. Case 2, $q \leqslant p$

In this situation, we must presume that n is a positive integer or zero. As before $F_n(z)$ obeys 7.4.1(2), which is of order $M = p + 2$, and there are $(p + 2)$ descending solutions of the form

$$
\mathscr{L}^{(\alpha_t)}_{p+2,q}(z) = \frac{(\alpha_p)^*_{-\alpha_t} z^{-\alpha_t}}{(\rho_q)_{-\alpha_t}}
$$

$$
\times {}_{q+1}F_{p+1}\left(\begin{matrix}\alpha_t, 1+\alpha_t-\rho_q \\ 1+\alpha_t+n, 1+\alpha_t-n-\lambda, 1+\alpha_t-\alpha_p^*\end{matrix}\middle| \frac{(-)^{q-p+1}}{z}\right),
$$

$$
t = 1, 2, \ldots, p, \tag{1}
$$

$$
\mathscr{H}^{(1)}_2(z) = \frac{(\alpha_p)_n}{(\rho_q)_n}(-z)^n \, {}_{q+1}F_{p+1}\left(\begin{matrix}-n, -n-\rho_q+1 \\ 1-2n-\lambda, 1-n-\alpha_p\end{matrix}\middle| \frac{(-)^{q-p+1}}{z}\right), \tag{2}
$$

$$\mathscr{H}_2^{(2)}(z) = \frac{(\alpha_p)_{-n-\lambda}}{(\rho_q)_{-n-\lambda}} z^{-n-\lambda}$$
$$\times {}_{q+1}F_{p+1}\left(\begin{matrix} n+\lambda, n+\lambda-\rho_q+1 \\ 1+2n+\lambda, 1+n+\lambda-\alpha_p \end{matrix} \middle| \frac{(-)^{q-p+1}}{z}\right) \tag{3}$$

The restriction 7 4 3(4) guarantees that the solutions (1)–(3) are linearly independent, but this requirement may be waived as in the case $q = p + 1$

In general, $F_n(z)$ is a linear combination of (1)–(3) In fact, under any particular set of restrictions on the parameters α_i and λ, $F_n(z)$ equals one and only one of these $p + 2$ solutions, e g ,

$$F_n(z) = \frac{(n+\lambda)_{-\alpha_m}}{(n+1)_{\alpha_m}} \mathscr{L}_{p+2\ q}^{(\alpha_m)}(z) \tag{4}$$

if some one α_m is a negative integer, $-\alpha_m \leqslant n$, λ is not a negative integer,

$$F_n(z) = \frac{1}{(n+1)_{n+\lambda}} \mathscr{H}_2^{(2)}(z), \tag{5}$$

if no α_i is a negative integer, λ is a negative integer, $2n > -\lambda$, etc We remark that although the $\mathscr{L}_{p+2\ q}^{(\alpha_i)}(z)$, $\mathscr{H}_2^{(1)}(z)$, and $\mathscr{H}_2^{(2)}(z)$ are descending series in z, the way n appears makes them suitable for computation for large n For the purposes of asymptotic equivalence for large n, one permits all solutions to appear, and writes for $q \leqslant p$,

$${}_{p+2}F_q\left(\begin{matrix} -n, n+\lambda\ \alpha_p \\ \rho_q \end{matrix} \middle| z\right) \sim \sum_{i=1}^{p} \frac{(n+\lambda)_{-\alpha_i}}{(n+1)_{\alpha_i}} \mathscr{L}_{p+2\ q}^{(\alpha_i)}(z)$$
$$+ (n+\lambda)_n \mathscr{H}_2^{(1)}(z) + \frac{1}{(n+1)_{n+\lambda}} \mathscr{H}_2^{(2)}(z), \tag{6}$$

where the connecting constants of the various solutions are those values which hold in the particular situation when $F_n(z)$ exactly equals that solution Then the dominant term of (6) under any set of conditions is that solution to which $F_n(z)$ is exactly equal As in our previous analysis, this actually determines the coefficients of the solutions (1) in (6) only asymptotically when no α_i is a negative integer or zero However, by an argument similar to that used for the case $q = p + 1$, see the discussion surrounding 7 4 3(13), one can determine these coefficients precisely and so write (6) as we did

If $p > q$, (6) is suitable for computational purposes for large n

However, if $p = q$, the (in general) dominant term (2) converges very slowly. If $p = q$, then by elementary series manipulation,

$$\begin{aligned}\mathscr{H}_2^{(1)}(z) &= \frac{(\alpha_p)_n}{(\rho_p)_n}(-z)^n\, {}_{p+1}F_{p+1}\left(\begin{matrix}-n, -n-\rho_p+1\\ 1-2n-\lambda, 1-n-\alpha_p\end{matrix}\middle| -\frac{1}{z}\right)\\ &= \frac{(\alpha_p)_n}{(\rho_p)_n}(-z)^n \exp\left\{-\frac{n(n+\rho_p-1)}{z(2n+\lambda-1)(n-1+\alpha_p)}\right\}\\ &\quad\times\left\{1-\frac{(n+\rho_p-1)}{8(2n+\lambda-1)(n-1+\alpha_p)z^2}+O(n^{-2}z^{-2})\right\}. \qquad (7)\end{aligned}$$

We note that (6) holds for all z except at the singular points zero and infinity as $n \to \infty$. For fixed n and variable z, it appears sufficient to require $|z| \geqslant O(1)$ to insure that the correction terms remain small.

Finally observe that by confluence,

$$\lim_{\sigma\to\infty} {}_{p+3}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p, \sigma\\ \rho_q\end{matrix}\middle| \frac{z}{\sigma}\right) = {}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p\\ \rho_q\end{matrix}\middle| z\right), \qquad (8)$$

and that confluences on the $\mathscr{L}_{p+2,q}^{(\alpha_t)}(z)$ terms in (6) can be carried out in a similar fashion. By $p+1-q$ such confluences, we drop from Case 2 to Case 1, and by comparison of the $\mathscr{L}_{p+2,p+1}^{(\alpha_t)}(z)$ terms, we see the consistency of our choice of the connecting constants A_t, $t = 1, 2,\ldots, p$, in 7.4.2(8).

7.4.5. Case 3, $q \geqslant p+2$

In this instance, we take

$$N^\beta = n(n+\lambda), \qquad \beta = q-p+1, \qquad \beta \geqslant 3. \qquad (1)$$

The analysis is similar to Case 1 ($\beta = 2$). $F_n(z)$ obeys 7.4.1(2), which is of order $M = p+\beta$. The $\mathscr{L}_{p+2,q}^{(\alpha_t)}(z)$ functions of 7.4.4(1) are p formal, descending series in z which satisfy 7.4.1(2). Although in general they are divergent, they serve as asymptotic expansions for large z to valid solutions of 7.4.1(2), and as mentioned before, are suitable for computation if n is large. The confluence argument at the end of 7.4.4 indicates that the $\mathscr{L}_{p+2,q}^{(\alpha_t)}(z)$ solutions and their connecting constants for $F_n(z)$ are independent of the case number. The lead terms of the exponential asymptotic expansions of the remaining $M-p = \beta$ solutions around infinity are computed by the formal procedure given in 7.4.1, and are denoted by $\mathscr{K}_\beta^{(j)}(z)$, $j = 1, 2,\ldots, \beta$. Thus under suitable restrictions on z,

$$F_n(z) \sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathscr{L}_{p+2,q}^{(\alpha_t)}(z) + \sum_{j=1}^{\beta} A_{p+j}\, \mathscr{K}_\beta^{(j)}(z). \qquad (2)$$

The constants A_{p+j}, $j = 1, 2, \ldots, \beta$, are determined asymptotically in their lead terms by exploitation of the fact that

$$\lim_{n\to\infty} F_n\left(\frac{z}{n(n+\lambda)}\right) = {}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| -z\right), \tag{3}$$

and that for $0 \ll |z| \ll n(n+\lambda)$, the representation (2) coalesces with the asymptotic representation of the latter ${}_pF_q$ for large z, see 5 11 2(1) Thus we write, through the τ_2 terms [see 7 4 1(6)],

$$\begin{aligned}
&{}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p\\ \rho_q\end{matrix}\middle| z\right)\\
&\sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathcal{L}^{(\alpha_t)}_{p+2\ q}(z) + \frac{2(2\pi)^{(1-\beta)/2}\Gamma(\rho_q)}{\beta^{1/2}\Gamma(\alpha_p)} (N^\beta z)^\gamma\\
&\quad\times \exp\{Nz^{1/\beta}\beta\cos(\pi/\beta) + az/3 - (Nz^{1/\beta})^{-1}\Omega(z)\cos(\pi/\beta) + O(N^{-2})\}\\
&\quad\times \cos\{Nz^{1/\beta}\beta\sin(\pi/\beta) + \pi\gamma + (Nz^{1/\beta})^{-1}\Omega(z)\sin(\pi/\beta) + O(N^{-2})\}\\
&\qquad + (\beta-2) \text{ exponentially lower order terms,}\\
&\qquad\qquad |\arg z| \leqslant \pi - \epsilon, \qquad \epsilon > 0,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&N^\beta = n(n+\lambda), \qquad \beta = q - p + 1,\\
&B_1 = \sum_{t=1}^{p} \alpha_t, \qquad C_1 = \sum_{t=1}^{q} \rho_t,\\
&B_2 = \sum_{s=2}^{p}\sum_{t=1}^{s-1} \alpha_s\alpha_t, \qquad C_2 = \sum_{s=2}^{q}\sum_{t=1}^{s-1} \rho_s\rho_t,\\
&\gamma = (2\beta)^{-1}(\beta - 1 + 2B_1 - 2C_1),\\
&\Omega(z) = (2\beta-1)^{-1}\lambda_1 z^2 + (\beta-1)^{-1}\lambda_2 z - \lambda_3,
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\lambda_1 = a/3,\\
&\lambda_2 = \lambda_1(3\lambda + 2B_1 - 2C_1 + 1) - b,\\
&\lambda_3 = C_2 - B_2 + (2\beta)^{-1}(B_1 - C_1)[\beta(B_1 + C_1) + B_1 - C_1 - 2]\\
&\qquad + (24\beta)^{-1}(\beta-1)(\beta-11)
\end{aligned}$$

$$a = 1 \text{ if } \beta = 3, \qquad a = 0 \text{ if } \beta \neq 3,$$

$$b = 1 \text{ if } \beta = 4, \qquad b = 0 \text{ if } \beta \neq 4$$

We point out that this procedure and hence (4) fails to include higher order terms in the connecting constants. In this regard, see the remarks surrounding 7.4.2(5–7).

A representation valid along the negative real axis can be similarly constructed. The connecting constants are determined from (4) by comparing the dominant terms in those overlapping regions where both representations are valid. Hence

$$
\begin{aligned}
&{}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ \rho_q\end{matrix}\middle| -z\right) \\
&\quad \sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathscr{L}^{(\alpha_t)}_{p+2,q}(ze^{i\delta\pi}) + \frac{(2\pi)^{(1-\beta)/2}\Gamma(\rho_q)}{\beta^{1/2}\Gamma(\alpha_p)}(N^\beta z)^\gamma \\
&\qquad \times \exp\{Nz^{1/\beta}\beta - (az/3) - (Nz^{1/\beta})^{-1}\Omega(-z) + O(N^{-2})\} \\
&\qquad + (\beta - 1) \quad \text{exponentially lower order terms,} \\
&\quad |\arg z| \leqslant \pi - \epsilon, \qquad \epsilon > 0, \qquad \delta = +(-) \qquad \text{if} \quad \arg z \leqslant (>) 0, \qquad (6)
\end{aligned}
$$

where the notations are as in (4), (5).

For fixed z, (4) and (6) hold for all z save at the singular points zero and infinity as $n \to \infty$. But for n fixed and z varying, it seems sufficient to have $|z| \geqslant O(n^{-2})$ near zero and $|z| \leqslant O(n^{\beta-2})$ near infinity to insure that the correction terms remain small.

7.4.6. Confluent Functions and Polynomials

By confluence, if n is a positive integer,

$$
\lim_{\lambda\to\infty} {}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ \rho_q\end{matrix}\middle| \frac{z}{n+\lambda}\right) = {}_{p+1}F_q\left(\begin{matrix}-n, \alpha_p \\ \rho_q\end{matrix}\middle| z\right). \qquad (1)
$$

From 3.5(21), we see that this relation is still valid if n is not a positive integer provided that $p + 1 < q$, $|z| < \infty$, or $p + 1 = q$, $|z| < |n + \lambda|$. Thus the asymptotic representation for the right-hand side of (1) may be derived from those for the above ${}_{p+2}F_q$ except when $p = q$, which can be treated directly by the methods used to get 7.4.2(8).

If $p = 0$ and $q = 1$, the right-hand side of (1) is essentially the Laguerre polynomial, see 8.1(33). If $q = p + 1$, we call (1) a generalized Laguerre polynomial, and in general we call (1) an extended Laguerre polynomial.

We first treat the case $q = p + 1$ explicitly. Replacing z by $z/(n + \lambda)$ corresponds to replacing θ by $2\{z/(n + \lambda)\}^{1/2}\{1 + O(\lambda^{-1})\}$ in 7.4.2(8).

Thus,

$$
\begin{aligned}
&{}_{p+1}F_{p+1}\left(\begin{matrix}-n, \alpha_p\\ \rho_{p+1}\end{matrix}\middle| z\right)\\
&\quad\sim \sum_{i=1}^{p} \frac{(\alpha_p)^*_{-\alpha_i} z^{-\alpha_i}}{(n+1)_{\alpha_i}(\rho_{p+1})_{-\alpha_i}}\, {}_{p+2}F_p\left(\begin{matrix}\alpha_i, 1+\alpha_i-\rho_{p+1}\\ 1+\alpha_i+n, 1+\alpha_i-\alpha_p^*\end{matrix}\middle| -\frac{1}{z}\right)\\
&\qquad+\frac{\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)\pi^{1/2}}(nz)^{\gamma}\exp\{(z/2)+(nz)^{-1}\psi_2(z)+O(n^{-2})\}\\
&\qquad\times\cos\{2(nz)^{1/2}+n\gamma-(nz)^{-1/2}\psi_1(z)-(nz)^{-3/2}\psi_3(z)+O(n^{-5/2})\},\\
&\qquad\qquad |\arg z| \leqslant \pi-\epsilon, \qquad \epsilon>0, \quad (2)
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_1(z) &= (12)^{-1}z^2+(2)^{-1}(B_1-C_1)z+\omega_1,\\
\psi_2(z) &= (16)^{-1}z^2+a_2^*z+\omega_2,\\
a_2^* &= (8)^{-1}(B_1-C_1)(2B_1+2C_1-1)+(2)^{-1}(C_2-B_2),\\
\psi_3(z) &= (320)^{-1}z^4+(48)^{-1}(B_1-C_1)z^3+\omega_3z^2+\omega_4z+\omega_5,\\
\omega_1 &= (4)^{-1}(B_1-C_1)(3B_1+C_1-2)+C_2-B_2-3/16,\\
\omega_2 &= (16)^{-1}(C_1-B_1)(8B_2-8B_1^2+11B_1+C_1-2)\\
&\quad+(4)^{-1}(C_2-B_2)(2B_1-3)-(2)^{-1}(C_3-B_3)+3/64,\\
\omega_3 &= (32)^{-1}(B_1-C_1)(2-B_1-3C_1)+(8)^{-1}(B_2-C_2)-1/128, \qquad (3)\\
\omega_4 &= (16)^{-1}(B_1-C_1)(8B_2-5B_1^2-C_1^2-2B_1C_1+6B_1+2C_1-3/4)\\
&\quad+(4)^{-1}(B_2-C_2)(B_1+C_1-2)-(2)^{-1}(B_3-C_3),\\
\omega_5 &= (192)^{-1}(B_1-C_1)(C_1^3+5B_1C_1^2+35B_1^2C_1-105B_1^3+236B_1^2\\
&\quad+160B_1B_2-24B_2C_1-8C_1C_2-40B_1C_1-4C_1^2-192B_2\\
&\quad-64B_3-291B_1/2-C_1/2+9)+(24)^{-1}(B_2-C_2)(2C_2-10B_2\\
&\quad+6C_1-30B_1-7B_1C_1+15B_1^2+73/4)\\
&\quad+(6)^{-1}(B_3-C_3)(C_1-3B_1+6)+(3)^{-1}(B_4-C_4)+21/1024,
\end{aligned}
$$

and the B_i's, C_i's, and γ are as in 7 4 2 (9)

Note that in the notation of 7 4 2(9) $\omega_1=\mu_3$, $\omega_2=\mu_5$, and $\omega_5=\mu_{10}/3$ Further, we can show that $-\omega_3$ is the coefficient of λ^2 in μ_8 and ω_4 is the coefficient of λ in μ_9

As usual (2) holds for all fixed z as $n\to\infty$ except at the singular points zero and infinity, and along the negative real axis If n is fixed and z is allowed to vary, additional restrictions must be put on z to insure that the correction terms remain small

Treating 7.4.2(10) in a similar fashion and using the same notations as in (2), we have

$$
\begin{aligned}
&{}_{p+1}F_{p+1}\left(\begin{matrix}-n,\ \alpha_p\\ \rho_{p+1}\end{matrix}\,\middle|\,-z\right)\\
&\quad\sim \sum_{t=1}^{p} \frac{(\alpha_p)^*_{-\alpha_t}(ze^{i\delta\pi})^{-\alpha_t}}{(n+1)_{\alpha_t}(\rho_{p+1})_{-\alpha_t}}\,{}_{p+2}F_p\left(\begin{matrix}\alpha_t,\ 1+\alpha_t-\rho_{p+1}\\ 1+\alpha_t+n,\ 1+\alpha_t-\alpha_p^*\end{matrix}\,\middle|\,\frac{1}{z}\right)\\
&\qquad+\frac{\Gamma(\rho_{p+1})}{\Gamma(\alpha_p)\pi^{1/2}}(nz)^{\gamma}\exp\{-(z/2)-(nz)^{-1}\psi_2(-z)+O(n^{-2})\}\\
&\qquad\times\cosh\{2(nz)^{1/2}+(nz)^{-1/2}\psi_1(-z)-(nz)^{-3/2}\psi_3(-z)+O(n^{-5/2})\},\\
&|\arg z|\leqslant\pi-\epsilon,\qquad \epsilon>0,\qquad \delta=+(-)\qquad\text{if}\quad \arg z\leqslant(>)\,0. \qquad (4)
\end{aligned}
$$

Again by confluence of 7.4.5(4, 6), (2), and (4), we can get asymptotic representations for ${}_pF_q(\alpha_p\,;\,\rho_q\,;\,z)$, $p<q$, for large z. These results are given by 5.11.4(2–5).

As remarked in the discussion following (1), results for the case $p=q$ are found by the same techniques employed to get 7.4.2(8). For the Darboux analysis, we employ 9.1(39). We suppose that n is a large positive integer, though the equations hold for more general n under suitable restrictions. Let

$$
\overline{\mathscr{P}}^{(\alpha_t)}_{p+1,p}(z)=\frac{(\alpha_p)^*_{-\alpha_t}z^{-\alpha_t}}{(\rho_p)_{-\alpha_t}}\,{}_{p+1}F_p\left(\begin{matrix}\alpha_t,\ 1+\alpha_t-\rho_p\\ \alpha_t+n+1,\ 1+\alpha_t-\alpha_p^*\end{matrix}\,\middle|\,\frac{1}{z}\right), \qquad (5)
$$

$$
\begin{aligned}
&\gamma=B_1-C_1,\qquad u=B_2-C_2+\gamma(1-B_1),\\
&v=C_2-B_2+\tfrac{1}{2}\gamma(B_1+C_1-1),
\end{aligned} \qquad (6)
$$

where the notation is as in 7.4.5(5) with $\beta=1$. We find that

$$
\begin{aligned}
{}_{p+1}F_p\left(\begin{matrix}-n,\ \alpha_p\\ \rho_p\end{matrix}\,\middle|\,z\right)&\sim\sum_{t=1}^{p}\frac{1}{(n+1)_{\alpha_t}}\overline{\mathscr{P}}^{(\alpha_t)}_{p+1,p}(z)\\
&\quad+\frac{\Gamma(\rho_p)}{\Gamma(\alpha_p)}(1-z)^n\left(\frac{nze^{-i\pi}}{1-z}\right)^{\gamma}\exp\left\{\frac{u}{nz}+\frac{v}{n}+O(n^{-2})\right\},\\
z\neq 0,&\qquad |\arg(1-z)|\leqslant\pi-\epsilon,\qquad \epsilon>0, \qquad (7)
\end{aligned}
$$

and the sign of $[nze^{-i\pi}/(1-z)]^{\gamma}$ is chosen so that this term is positive if γ is real and $z<0$. Here z can take on unity if it is approached from the left. Equation (7) is also valid for $|\arg(z-1)|\leqslant\pi-\epsilon$, $\epsilon>0$, provided in the second term of (7) we replace $[nze^{-i\pi}/(1-z)]^{\gamma}$ by $[nz/(z-1)]^{\gamma}$ and choose the sign of the latter so that it is positive if γ

is real and $z > 1$ In this situation z can take on the value unity if it is approached from the right Further,

$$
{}_{p+1}F_p\left(\begin{matrix}-n, \alpha_p \\ \rho_p\end{matrix}\middle| -z\right) \sim \sum_{i=1}^{p} \frac{1}{(n+1)_{\alpha_i}} \overline{\mathscr{P}}^{(\alpha_i)}_{p+1\ p}(ze^{i\pi\delta})
$$

$$
+ \frac{\Gamma(\rho_p)}{\Gamma(\alpha_p)}(1+z)^n \left(\frac{nz}{1+z}\right)^{\gamma} \exp\left\{-\frac{u}{nz} + \frac{v}{n} + O(n^{-2})\right\},
$$

$$
z \neq 0, \qquad |\arg z| \leqslant \pi - \epsilon, \qquad \epsilon > 0,
$$

$$
\delta = +(-) \qquad \text{if} \quad \arg z < (>) 0 \tag{8}
$$

n (8), $[z/(1+z)]^{\gamma}$ is positive if $z > 0$ and γ is real

Chapter VIII ORTHOGONAL POLYNOMIALS

8.1. Orthogonal Properties

In virtually all of our work, we are concerned with the classical orthogonal polynomials. However, to introduce the concept of orthogonality and its applications, it is convenient to consider real functions defined over real intervals, although the ideas are readily extended to complex functions defined over paths in the complex plane. See the remark following (34).

The standard book on the subject is by Szegö (1959). For other references, see Kacmarz and Steinhaus (1951), Erdélyi *et al.* (1953, Vol. 2, Chapter 10), Tricomi (1955), Sansone (1959), Geronimus (1960), and Abramowitz and Stegun (1964).

Consider an interval (a, b) and a weight function $w(x)$ which is nonnegative over this interval. Let $\{\theta_n(x)\}$ be a sequence of functions, such that $w\theta_n^2$ is integrable in (a, b). We define the scalar product.

$$(\theta_n, \theta_m) = \int_a^b w(x)\, \theta_n(x)\, \theta_m(x)\, dx. \tag{1}$$

If $a(x)$ is a nondecreasing function, we may generalize (1) by the Stieltjes integral

$$(\theta_n, \theta_m) = \int_a^b \theta_n(x)\, \theta_m(x)\, da(x). \tag{2}$$

So if $a(x)$ is absolutely continuous, (2) becomes (1) with $w(x) = a'(x)$. If $a(x)$ is a jump function which is constant except for jumps w_i at $x = x_i$, then (2) reduces to

$$(\theta_n, \theta_m) = \sum_{i=0}^{n} w_i \theta_n(x_i)\, \theta_m(x_i). \tag{3}$$

This is the appropriate definition for the scalar product for functions of a discrete variable.

The sequence of functions $\{\theta_n(x)\}$ is said to be orthogonal [with respect to a weight function $w(x)$] over the interval (a, b) if

$$(\theta_n, \theta_m) = h_n \delta_{mn} \tag{4}$$

where

$$h_n = (\theta_n, \theta_n) = \int_a^b w(x)[\theta_n(x)]^2\, dx, \tag{5}$$

and δ_{mn} is the Kronecker delta function That is,

$$\begin{aligned} \delta_{mn} &= 0 \quad \text{if} \quad m \neq n \\ &= 1 \quad \text{if} \quad m = n \end{aligned} \tag{6}$$

If $h_n = 1$ for all n, the system is said to be orthonormal Clearly any orthogonal system can be made orthonormal if we replace $\theta_n(x)$ by $\theta_n(x)/h_n^{1/2}$

A natural question concerns the possibility of representing an arbitrary function $f \equiv f(x)$ as a sum of orthogonal functions, thus

$$f = \sum_{k=0}^{\infty} a_k \theta_k \tag{7}$$

Assuming this is so, then on a formal basis we have

$$(f, \theta_k) = \sum_{n=0}^{\infty} a_n(\theta_k, \theta_n) = a_k h_k, \qquad a_k = h_k^{-1}(f, \theta_k) \tag{8}$$

The a_k's are often called the Fourier coefficients associated with f, and the series on the right of (7) is called a generalized Fourier series The fact that we can calculate a_k does not guarantee that the series on the right of (7) converges, or if the series converges that its sum is f Throughout the discussion we suppose that $\int_a^b f^2(x)\, w(x)\, dx$ exists and is finite in the Lebesgue sense The class of functions for which this is true is called L_w^2 We further suppose that $\{\theta_n(x)\}$ belongs to this class Let

$$f_n = \sum_{k=0}^{n} b_k \theta_k \tag{9}$$

We call f_n an approximation to f of order n A measure of the accuracy of this approximation is afforded by the integral

$$I_n(b_k) \equiv I_n(b_0 \quad , b_n) = \int_a^b \left[f(x) - \sum_{k=0}^{n} b_k \theta_k(x) \right]^2 w(x)\, dx \tag{10}$$

A best choice for the b_k's is that which makes $I_n(b_k)$ a minimum, if such exists, in which event we speak of a best mean square approximation to $f(x)$ We now prove the following theorem

Theorem 1. *Of all the* nth *order approximations to* $f(x)$, *the best in the mean square sense occurs when* $b_k = a_k$.

In other words, the best approximation is the $(n+1)$th partial sum of the series on the right-hand side of (7).

PROOF. We have

$$I_n(b_h) = \int_a^b f^2(x)\, w(x)\, dx - 2 \sum_{k=0}^{n} b_k \int_a^b f(x)\, \theta_k(x)\, w(x)\, dx$$

$$+ \sum_{k=0}^{n} b_k^2 \int_a^b \theta_k^2(x)\, w(x)\, dx + \sum_{\substack{k,m=0 \\ k \neq m}}^{n} b_k b_m \int_a^b \theta_k(x)\, \theta_m(x)\, w(x)\, dx.$$

Use the orthogonality property, and (8). Then

$$I_n(b_h) = \int_a^b f^2(x)\, w(x)\, dx - 2 \sum_{k=0}^{n} a_k b_k h_k + \sum_{k=0}^{n} b_k^2 h_k$$

$$= \int_a^b f^2(x)\, w(x)\, dx + \sum_{k=0}^{n} (b_k - a_k)^2\, h_k - \sum_{k=0}^{n} a_k^2 h_k\,,$$

and this is least when $b_k = a_k$. Then a measure of the accuracy of the approximation is given by

$$I_n(a_h) = \int_a^b f^2(x)\, w(x)\, dx - \sum_{k=0}^{n} a_k^2 h_k\,. \tag{11}$$

Since $I_n(a_h) \geqslant 0$, $\sum_{k=0}^{n} a_k^2 h_k$ converges as $n \to \infty$, and we have Bessel's inequality

$$\sum_{k=0}^{\infty} a_k^2 h_k \leqslant \int_a^b f^2(x)\, w(x)\, dx. \tag{12}$$

When there is equality (the formula then goes by the name of Parseval) for every function in L_w^2, then $\{\theta_n(x)\}$ is said to be closed in L_w^2. In this case

$$\lim_{n\to\infty} \int_a^b \left[f(x) - \sum_{k=0}^{n} a_k \theta_k(x) \right]^2 w(x)\, dx = 0, \tag{13}$$

and the partial sums of the generalized Fourier series are said to converge in the mean to $f(x)$.

In the case of functions of a discrete variable [see the discussion around (3)] suppose that

$$f_n(x) = \sum_{k=0}^{n} c_k \theta_k(x), \tag{14}$$

$$\sum_{i=0}^{n} \theta_k(x_i)\, \theta_m(x_i)\, W_i = H_m \delta_{km} \tag{15}$$

where $W_i = W(x_i)$ is positive and δ_{km} is the Kronecker delta function Now multiply both sides of (14) by $\theta_m(x_i)\, W_i$ and sum on i from 0 to n Apply (15) Then

$$c_k = H_k^{-1} \sum_{i=0}^{n} \theta_k(x_i) f_n(x_i)\, W_i \tag{16}$$

Thus, if $f(x)$ is known at the $(n + 1)$ distinct points x_i, $i = 0, 1, \ldots, n$, $f(x_i) = f_n(x_i)$, then $f_n(x)$ as given by (14) is a curve fit to $f(x)$

Suppose we have another curve fit to $f(x)$ in the form

$$f_n^*(x) = \sum_{k=0}^{n} d_k \theta_k(x) \qquad f_n^*(x_i) = f_n(x_i) = f(x_i) \tag{17}$$

Then a measure of the accuracy of the curve fitting process may be described by

$$I(d_k) = \sum_{i=0}^{n} \left\{ f_n(x_i) - \sum_{k=0}^{n} d_k \theta_k(x_i) \right\}^2 W_i \tag{18}$$

After the manner of deriving Theorem 1 we have the following

Theorem 2 *The best approximation to $f_n(x)$ in the sense that $I_n(d_k)$ is least happens when $c_k = d_k$, and in this instance*

$$I_n(c_k) = \sum_{i=0}^{n} W_i f_n^2(x_i) - \sum_{k=0}^{n} h_k c_k^2 \tag{19}$$

We omit the proof Such an approximation is said to be best in the sense of least squares

Next we consider orthogonal polynomials for which we use the symbol $q_n(x)$ They possess some important properties (given by Theorems 3 5 below) which make them very suitable for use in approximation theory

Theorem 3. *The zeros of $q_n(x)$ are simple and lie in the interior of $[a, b]$.*

PROOF. Now any polynomial of degree $m < n$ is a linear sum of the polynomials $q_i(x)$, $i = 0, 1,..., m$, and so is orthogonal to $q_n(x)$. If $q_n(x) = 0$ for $x = b_i$, $i = 1, 2,..., m < n$, $b_i \in (a, b)$, then

$$p(x) = \prod_{i=1}^{m} (x - b_i)\, q_n(x)$$

is one-signed throughout $[a, b]$. This would imply that

$$\int_a^b w(x)\, p(x)\, dx \neq 0.$$

However, in view of orthogonality, this can happen only if $m = n$.

Theorem 4. *Let*

$$f(x) = \sum_{k=0}^{\infty} a_k q_k(x), \qquad f_n(x) = \sum_{k=0}^{n} a_k q_k(x).$$

Then $[f(x) - f_n(x)]$ vanishes at least $(n + 1)$ times in $[a, b]$.

The idea of the proof is similar to that for Theorem 3 and we omit details.

Theorem 5. *Any three consecutive orthogonal polynomials satisfy a recurrence formula of the form*

$$q_{n+1}(x) = (A_n x + B_n)\, q_n(x) - C_n q_{n-1}(x), \qquad n = 1, 2,\ldots . \tag{20}$$

Furthermore, with

$$q_n(x) = \sum_{k=0}^{n} a_{k,n} x^k,$$

we prove that

$$\begin{aligned} A_n &= \frac{a_{n+1,n+1}}{a_{n,n}}, \qquad B_n = A_n(r_{n+1} - r_n), \qquad r_n = \frac{a_{n-1,n}}{a_{n,n}}, \\ C_n &= \frac{A_n h_n}{A_{n-1} h_{n-1}} = \frac{a_{n-1,n+1} a_{n-1,n-1} h_n}{(a_{n,n})^2\, h_{n-1}}, \qquad h_n = (q_n, q_n). \end{aligned} \tag{21}$$

PROOF. First we note that with A_n defined as in (21), $q_{n+1}(x) - A_n x q_n(x)$ is a polynomial of degreee n or less, and of necessity, we must have

$$q_{n+1}(x) - A_n x q_n(x) = \sum_{k=0}^{n} c_k q_k(x).$$

Multiply both sides of the last equation by $q_m(x)\,w(x)$ and integrate from a to b Then by orthogonality, we see that $c_0 = c_1 = \quad = c_{n-2} = 0$ and

$$-A_n(xq_{n-1}\,,\,q_n) = c_{n-1}h_{n-1}\,. \tag{22}$$

Thus,

$$q_{n+1}(x) - A_n x q_n(x) = c_{n-1}q_{n-1}(x) + c_n q_n(x)$$

which proves (20) when we identify $c_{n-1} = -C_n\,,\ c_n = B_n$ Now

$$xq_{n-1}(x) - \frac{a_{n-1\ n-1}}{a_{n\ n}}\,q_n(x) = \sum_{k=0}^{n-1} d_k q_k(x)$$

since the polynomial on the left is of degree $(n-1)$ or less Multiply both sides of the latter by $q_n(x)\,w(x)$, integrate from a to b to get

$$c_{n-1} = -C_n = -A_n h_n / A_{n-1} h_{n-1} \tag{23}$$

Finally, the value of B_n follows upon equating like coefficients of x^n in (20) Note that (20) is also valid for $n = 0$ provided we put $q_{-1}(x) = 0$

From (20), we easily obtain the Christoffel–Darboux formulas

$$\sum_{k=0}^{n} h_k^{-1} q_k(x)\,q_k(y) = (A_n h_n)^{-1}\,\frac{q_{n+1}(x)\,q_n(y) - q_n(x)\,q_{n+1}(y)}{x-y}\,,$$

$$\sum_{k=0}^{n} h_k^{-1} q_k^2(x) = (A_n h_n)^{-1}\,[q_n(x)\,q_{n+1}(x) - q_n(x)\,q_{n+1}(x)] \tag{24}$$

Orthogonal polynomials also play an important role in numerical integration This topic is considered in 16 3 1

As previously remarked we are primarily interested in the classical orthogonal polynomials These polynomials are special cases of Gaussian or confluent hypergeometric functions and virtually all the results given in the later sections can be derived from the material in Chapters III and IV The data in Chapters VII and IX are also pertinent From (25)–(34), we see that all the classical orthogonal polynomials stem from the Jacobi polynomial $P_n^{(\alpha\ \beta)}(x)$ of degree n If, in the hypergeometric representation of $P_n^{(\alpha\ \beta)}(x)$, the parameter n is allowed to be an arbitrary complex number, then the hypergeometric function is called a Jacobi function of the first kind The Jacobi function of the second kind is essentially the second solution of the differential equation satisfied by the function of the first kind Similarly, we speak of Legendre functions

TABLE 8.1

Name	a	b	$w(x)$	Eq.
Jacobi:				
$P_n^{(\alpha,\beta)}(x) = \dfrac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\middle\vert\, \dfrac{1-x}{2}\right)$	-1	1	$(1-x)^\alpha(1+x)^\beta$	(25)
Jacobi (shifted):				
$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$	0	1	$(1-x)^\alpha x^\beta$	(26)
Gegenbauer or ultraspherical:				
$C_n^{(\alpha+1/2)}(x) = \dfrac{(2\alpha+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\alpha)}(x)$	-1	1	$(1-x^2)^\alpha$	(27)
Legendre:				
$P_n(x) = P_n^{(0,0)}(x)$	-1	1	1	(28)
Chebyshev (first kind):				
$T_n(x) = \dfrac{n!}{(1/2)_n} P_n^{(-1/2,-1/2)}(x)$	-1	1	$(1-x^2)^{-1/2}$	(29)
Chebyshev (first kind, shifted):				
$T_n^*(x) = T_n(2x-1)$	0	1	$[x(1-x)]^{-1/2}$	(30)
Chebyshev (second kind):				
$U_n(x) = \dfrac{(n+1)!}{(3/2)_n} P_n^{(1/2,1/2)}(x)$	-1	1	$(1-x^2)^{1/2}$	(31)
Chebyshev (second kind, shifted):				
$U_n^*(x) = U_n(2x-1)$	0	1	$[x(1-x)]^{1/2}$	(32)
Laguerre:				
$L_n^{(\alpha)}(x) = \lim\limits_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\dfrac{2x}{\beta}\right)$	0	∞	$e^{-x}x^\alpha$	
$= \lim\limits_{\beta\to\infty} R_n^{(\alpha,\beta)}\left(1-\dfrac{x}{\beta}\right)$				
$= (-)^n \lim\limits_{\beta\to\infty} R_n^{(\beta,\alpha)}(x/\beta),$				
$L_n^{(\alpha)}(x) = \dfrac{(\alpha+1)_n}{n!}\,{}_1F_1(-n;\, \alpha+1;\, x)$				(33)
The polynomial introduced by Laguerre is the case $\alpha = 0$.				
Hermite:				
$H_{2m+\epsilon}(x) = (-)^m 2^{2m+\epsilon} m!\, x^\epsilon L_m^{(\epsilon-1/2)}(x^2),$ $\epsilon = 0$ or $\epsilon = 1$	$-\infty$	∞	e^{-x^2}	(34)

Though numerous properties of Jacobi polynomials carry over to Jacobi functions, we delineate results for the polynomials only

The classical orthogonal polynomials may be briefly described by Table 8 1 which gives the name of the polynomial, its analytical name, the parameters describing the range of integration, and the weight function In addition, data are given to show that all the polynomials are special cases of the Jacobi polynomial Identification of the Jacobi polynomial and the Laguerre polynomial in terms of a ${}_2F_1$ and ${}_1F_1$, respectively, is also presented

The Bessel polynomials ${}_2F_0(-n, n+\nu, 1/z)$, which are confluent forms of the Jacobi polynomials, form an orthogonal set where the integration is performed over complex paths [see 14 2(26)]

8.2. Jacobi Polynomials

The Jacobi polynomial has been defined by 8 1(25) We suppose that $\alpha > -1, \beta > -1$ so that $w(x)$ is nonnegative and integrable in $[-1, 1]$ However, many of the formal results are valid without this restriction. The term $\alpha + \beta + 1$ occurs very frequently and for simplification we put

$$\lambda = \alpha + \beta + 1 \tag{1}$$

In hypergeometric form, we have

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\lambda\\ \alpha+1\end{matrix}\middle|\,\frac{1-x}{2}\right) \tag{2}$$

$$= \frac{(-)^n(\beta+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\lambda\\ \beta+1\end{matrix}\middle|\,\frac{1+x}{2}\right) \tag{3}$$

Clearly,

$$P_n^{(\alpha,\beta)}(-x) = (-)^n P_n^{(\beta,\alpha)}(x),$$
$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!} \qquad P_n^{(\alpha,\beta)}(-1) = \frac{(-)^n(\beta+1)_n}{n!} \tag{4}$$

Also

$$P_n^{(\alpha,-1/2)}(2x^2-1) = \frac{(2n)!\,(\alpha+1)_n}{n!\,(\alpha+1)_{2n}} P_{2n}^{(\alpha,\alpha)}(x), \tag{5}$$

$$P_n^{(\alpha,1/2)}(2x^2-1) = \frac{(2n+1)!\,(\alpha+1)_n}{n!\,(\alpha+1)_{2n+1}} x^{-1} P_{2n+1}^{(\alpha,\alpha)}(x) \tag{6}$$

Thus $P_n^{(\alpha,\alpha)}(x)$ is an even (odd) polynomial if n is even (odd)

Rodrigues' formula is

$$2^n n!\, P_n^{(\alpha,\beta)}(x) = (-)^n (1-x)^{-\alpha}(1+x)^{-\beta}\,(d^n/dx^n)\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\}, \tag{7}$$

and from this we have

$$P_n^{(\alpha,\beta)}(x) = 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k. \tag{8}$$

Also,

$$P_n^{(\alpha,\beta)}(x) = \sum_{r=0}^{n} k_{r,n}^{(\alpha,\beta)} x^r,$$

$$\begin{aligned} k_{r,n}^{(\alpha,\beta)} &= \frac{(-)^r(\alpha+1)_n(-n)_r(n+\lambda)_r}{r!\,(\alpha+1)_r\,2^r n!}\,{}_2F_1\left(\begin{matrix} r-n,\, n+\lambda+r \\ \alpha+1+r \end{matrix}\,\middle|\,\frac{1}{2}\right) \\ &= \frac{(-)^n(\beta+1)_n(-n)_r(n+\lambda)_r}{r!\,(\beta+1)_r\,2^r n!}\,{}_2F_1\left(\begin{matrix} r-n,\, n+\lambda+r \\ \beta+1+r \end{matrix}\,\middle|\,\frac{1}{2}\right). \end{aligned} \tag{9}$$

In particular,

$$k_{n,n}^{(\alpha,\beta)} = \frac{(n+\lambda)_n}{2^n n!}, \qquad k_{n-1,n}^{(\alpha,\beta)} = \frac{(\alpha-\beta)\,\Gamma(2n+\lambda-1)}{2^n(n-1)!\,\Gamma(n+\lambda)}. \tag{10}$$

In general a more simple expression for (9) is not known unless $\alpha = \beta$.

$$k_{r,n}^{(\alpha,\alpha)} = \frac{(-)^n(\alpha+1)_n(-n)_r(n+2\alpha+1)_r\,\Gamma(\alpha+1+r)\,\Gamma(\frac{1}{2})}{2^r r!\,(\alpha+1)_r\,\Gamma[(r-n+1)/2]\,\Gamma[(r+n)/2+\alpha+1]\,n!} \tag{11}$$

and (11) vanishes whenever $n-r$ is an odd positive integer.

We have the following differential–difference properties:

$$(1-x^2)\frac{d^2P_n^{(\alpha,\beta)}(x)}{dx^2} + [\beta-\alpha-(\lambda+1)x]\frac{dP_n^{(\alpha,\beta)}(x)}{dx} + n(n+\lambda)\,P_n^{(\alpha,\beta)}(x) = 0. \tag{12}$$

$$\begin{aligned}(2n+\lambda-1)(1-x^2)\frac{dP_n^{(\alpha,\beta)}(x)}{dx} &= n[(\alpha-\beta)-(2n+\lambda-1)x]\,P_n^{(\alpha,\beta)}(x) \\ &\quad + 2(n+\alpha)(n+\beta)\,P_{n-1}^{(\alpha,\beta)}(x).\end{aligned} \tag{13}$$

$$\begin{aligned} &2(n+1)(n+\lambda)(2n+\lambda-1)\,P_{n+1}^{(\alpha,\beta)}(x) \\ &\quad = (2n+\lambda)[(2n+\lambda-1)(2n+\lambda+1)x+\alpha^2-\beta^2]\,P_n^{(\alpha,\beta)}(x) \\ &\qquad - 2(n+\alpha)(n+\beta)(2n+\lambda+1)\,P_{n-1}^{(\alpha,\beta)}(x). \end{aligned} \tag{14}$$

$$(2n+\lambda+1)(1-x)\,P_n^{(\alpha+1,\,\beta)}(x) = 2(n+\alpha+1)\,P_n^{(\alpha,\,\beta)}(x) - 2(n+1)\,P_{n+1}^{(\alpha,\,\beta)}(x) \tag{15}$$

$$(2n+\lambda+1)(1+x)\,P_n^{(\alpha,\,\beta+1)} = 2(n+\beta+1)\,P_n^{(\alpha,\,\beta)}(x) + 2(n+1)\,P_{n+1}^{(\alpha,\,\beta)}(x) \tag{16}$$

$$(1-x)\,P_n^{(\alpha+1,\,\beta)}(x) + (1+x)\,P_n^{(\alpha,\,\beta+1)}(x) = 2P_n^{(\alpha,\,\beta)}(x). \tag{17}$$

$$(2n+\lambda-1)\,P_n^{(\alpha-1,\,\beta)}(x) = (n+\lambda-1)\,P_n^{(\alpha,\,\beta)}(x) - (n+\beta)\,P_{n-1}^{(\alpha,\,\beta)}(x) \tag{18}$$

$$(2n+\lambda-1)\,P_n^{(\alpha,\,\beta-1)}(x) = (n+\lambda-1)\,P_n^{(\alpha,\,\beta)}(x) + (n+\alpha)\,P_{n-1}^{(\alpha,\,\beta)}(x) \tag{19}$$

$$P_n^{(\alpha,\,\beta-1)}(x) - P_n^{(\alpha-1,\,\beta)}(x) = P_{n-1}^{(\alpha,\,\beta)}(x) \tag{20}$$

$$2^m \frac{d^m P_n^{(\alpha,\,\beta)}(x)}{dx^m} = (n+\lambda)_m\, P_{n-m}^{(\alpha+m,\,\beta+m)}(x) \qquad m = 1, 2, \ldots, n \tag{21}$$

The orthogonality property is given by

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta\, P_n^{(\alpha,\,\beta)}(x)\, P_m^{(\alpha,\,\beta)}(x)\, dx = h_n \delta_{mn},$$

$$h_n = \frac{2^\lambda \Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\lambda)\,n!\,\Gamma(n+\lambda)} \tag{22}$$

Some integrals involving the Jacobi polynomial follow

$$\begin{aligned}\int_{-1}^{x} P_n^{(\alpha,\,\beta)}(t)\,dt &= 2\left\{\frac{(n+\lambda)}{(2n+\lambda+1)(2n+\lambda)} P_{n+1}^{(\alpha,\,\beta)}(x) + \frac{(\alpha-\beta)}{(2n+\lambda+1)(2n+\lambda-1)} P_n^{(\alpha,\,\beta)}(x)\right.\\ &\quad \left. - \frac{(n+\beta)(n+\alpha)}{(2n+\lambda)(2n+\lambda-1)(n+\lambda-1)} P_{n-1}^{(\alpha,\,\beta)}(x)\right\} + \frac{2(-)^n\,\Gamma(n+\beta+1)}{(n+\lambda-1)(n+1)!\,\Gamma(\beta)}\end{aligned} \tag{23}$$

$$\begin{aligned}2n\int_0^x (1-t)^\alpha (1+t)^\beta\, P_n^{(\alpha,\,\beta)}(t)\,dt &= P_{n-1}^{(\alpha+1,\,\beta+1)}(0) - (1-x)^{\alpha+1}(1+x)^{\beta+1}\, P_{n-1}^{(\alpha+1,\,\beta+1)}(x)\end{aligned} \tag{24}$$

$$\begin{aligned}2n\int_0^x (1-t^2)^\alpha\, P_n^{(\alpha,\,\alpha)}(t)\,dt &= \frac{(-)^{n-1}(\alpha+2)_{n-1}\,\Gamma(\alpha+2)\,\Gamma(\tfrac{1}{2})}{(n-1)!\,\Gamma[(2-n)/2]\,\Gamma[(n+3+2\alpha)/2]} - (1-x^2)^\alpha\, P_{n-1}^{(\alpha+1,\,\alpha+1)}(x)\end{aligned} \tag{25}$$

$$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)\,dx$$

$$= (-)^n \int_{-1}^{1} (1-x)^{\sigma}(1+x)^{\rho} P_n^{(\alpha,\beta)}(x)\,dx$$

$$= \frac{(-)^n (\beta+1)_n\, 2^{\rho+\sigma+1}\Gamma(\rho+1)\,\Gamma(\sigma+1)}{n!\,\Gamma(\rho+\sigma+2)}\, {}_3F_2\left(\begin{matrix}-n, n+\lambda, \sigma+1\\ \beta+1, \rho+\sigma+2\end{matrix}\middle|\, 1\right),$$

$$R(\rho) > -1, \qquad R(\sigma) > -1. \tag{26}$$

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)$$

$$= \frac{(-)^n\, 2^{\alpha+\sigma+1}\Gamma(\sigma+1)\,\Gamma(n+\alpha+1)(\beta-\sigma)_n}{n!\,\Gamma(\alpha+\sigma+n+2)},$$

$$R(\alpha) > -1, \qquad R(\sigma) > -1. \tag{27}$$

For other integrals involving Jacobi polynomials, see Erdélyi *et al.* (1954, Chapter 16). See also 3.6(26), 5.6.4(13, 17).

For the expansion of integral powers of x in series of Jacobi polynomials, we have

$$x^m = \sum_{n=0}^{m} a_n P_n^{(\alpha,\beta)}(x),$$

$$a_n = \frac{2^n m!\,\Gamma(n+\lambda)}{(m-n)!\,\Gamma(2n+\lambda)}\, {}_2F_1\left(\begin{matrix}n-m, \alpha+n+1\\ 2n+\lambda+1\end{matrix}\middle|\, 2\right), \tag{28}$$

$$x^{2m+\epsilon} = \sum_{n=0}^{m} b_n P_{2n+\epsilon}^{(\alpha,\alpha)}(x), \qquad \epsilon = 0 \quad \text{or} \quad \epsilon = 1,$$

$$b_n = \frac{(2m+\epsilon)!\,(2n+\alpha+\frac{1}{2}+\epsilon)\,\Gamma(2n+2\alpha+1+\epsilon)\,\Gamma(\frac{1}{2})}{2^{2m+2\alpha+\epsilon}(m-n)!\,\Gamma(2n+\alpha+\epsilon+1)\,\Gamma(m+n+\alpha+\frac{3}{2}+\epsilon)}. \tag{29}$$

Also,

$$(1-x)^m = \sum_{n=0}^{m} c_n P_n^{(\alpha,\beta)}(x),$$

$$c_n = \frac{(-)^n 2^m m!\,\Gamma(\alpha+m+1)(2n+\lambda)\,\Gamma(n+\lambda)}{(m-n)!\,\Gamma(\alpha+n+1)\,\Gamma(m+n+\lambda+1)}. \tag{30}$$

See 8.4 for expansions of arbitrary functions in series of Jacobi polynomials.

We have the generating functions

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, z^n = 2^{\alpha+\beta} R^{-1} (1 - z + R)^{-\alpha} (1 + z + R)^{-\beta},$$

$$R = (1 - 2xz + z^2)^{1/2}, \quad -1 < x < 1, \quad |z| < 1, \quad R = 1 \text{ if } z = 0, \tag{31}$$

$$\sum_{n=0}^{\infty} [(\lambda)_n/(\beta+1)_n]\, P_n^{(\alpha,\beta)}(x)\, z^n = (1+z)^{-\lambda}\, {}_2F_1\left(\begin{matrix}\lambda/2, (\lambda+1)/2 \\ \beta+1\end{matrix} \middle| \frac{2(x+1)z}{(1+z)^2}\right),$$

$$-1 < x < 1, \qquad |z| < 1 \tag{32}$$

Szego (1933) has shown that there exist functions $f_{m,\nu}(\theta)$ which are regular in $0 \leqslant \theta < \pi$ so that

$$P_n^{(\alpha,\beta)}(\cos\theta) \sim (\sin\theta/2)^{-\alpha} (\cos\theta/2)^{-\beta} (\theta \sin\theta)^{1/2} \sum_{m=0}^{\infty} \theta^m \left\{ \sum_{\nu=0}^{m} f_{m,\nu}(\theta)\, J_{n+\nu,m}(n\theta) \right\},$$

$$0 < \theta \leqslant \pi - \epsilon, \qquad 0 < \epsilon < \pi \tag{33}$$

where

$$n = n + \lambda/2, \qquad J_{\nu,m}(z) = \sum_{h=0}^{m} \binom{m}{h} J_{m-2h+\nu}(z) \tag{34}$$

The first few of the $f_{m,\nu}$'s are as follows

$$f_{0,0} = 1, \qquad f_{1,0}(\theta) = -\frac{\beta}{2}\left(\frac{\tan\theta/2}{2\theta}\right)^{1/2},$$

$$f_{1,1}(\theta) = -\frac{\alpha}{2\theta}\left[\left(\frac{\theta}{2\tan\theta/2}\right)^{1/2} - 1\right]$$

$$\begin{aligned} f_{2,0} = {} & \frac{\theta\cos\theta - \sin\theta}{32\theta^2 \sin\theta} + \frac{\alpha}{8}\,\frac{(\theta/2)\cos\theta/2 - \sin\theta/2}{\theta^2 \sin\theta/2} \\ & - \frac{\beta}{16}\,\frac{\tan\theta/2}{\theta} + \frac{\beta(\beta+1)}{16}\,\frac{\tan\theta/2}{\theta}, \end{aligned} \tag{35}$$

$$f_{2,1} = \frac{\alpha\beta}{8\theta}\left[1 - \left(\frac{2\tan\theta/2}{\theta}\right)^{1/2}\right],$$

$$f_{2,2}(\theta) = \frac{\alpha(\alpha+1)}{8\theta^2}\left[\left(\frac{\theta}{2\tan\theta/2}\right)^{1/2} - 1\right]^2$$

Further, if the series in (33) is truncated after $m = p - 1$ terms, $p \geqslant 1$, and the error in the Jacobi polynomial is notated R_p, then

$$\begin{aligned} R_p &= \theta^{p-\sigma_p-\alpha} O(n^{-\sigma_p}) \qquad \text{if} \quad n\theta \geqslant c, \\ &= O(n^{\alpha-p}) \qquad\qquad \text{if} \quad n\theta \leqslant c, \\ \sigma_p &= \tfrac{1}{2} + [\tfrac{1}{2}(p+1)], \end{aligned} \tag{36}$$

uniformly in $0 < \theta \leqslant \pi - \epsilon$, $0 < \epsilon < \pi$. Here c is a fixed positive number.

For the ultraspherical case ($\alpha = \beta$), with $\mu = \alpha + \frac{1}{2}$, we have

$$P_n^{(\alpha,\alpha)}(\cos\theta) \sim \frac{\Gamma(\frac{1}{2})\, 2^{1/2-\mu}}{\Gamma(\mu)} \left(\frac{\theta}{\sin\theta}\right)^{\mu} \sum_{m=0}^{\infty} \frac{f_m(\theta)\, \theta^{m-\mu+1/2} J_{\mu-m-1/2}[(n+\mu)\,\theta]}{(n+\mu)^{m-\mu+1/2}},$$

$$0 < \theta \leqslant \pi - \epsilon, \qquad 0 < \epsilon < \pi, \tag{37}$$

$$f_0(\theta) = 1, \qquad f_1(\theta) = \frac{\mu(\mu-1)}{2} \frac{(\theta\cos\theta - \sin\theta)}{\theta^2 \sin\theta},$$

$$f_2(\theta) = \frac{\mu(\mu-1)(\mu-2)}{8} \tag{38}$$

$$\times \frac{[(\mu-3)\sin^2\theta - 2(\mu-1)\,\theta\sin\theta\cos\theta + (\mu+1)\,\theta^2\cos^2\theta + \frac{4}{3}\theta^2\sin^2\theta]}{\theta^4 \sin^2\theta},$$

$$\begin{aligned} R_p &= \theta^{p-\mu} O(n^{\mu-p-1}) \qquad \text{if} \quad n\theta \geqslant c \\ &= O(n^{2\mu-2p-1}) \qquad\qquad \text{if} \quad n\theta \leqslant c, \end{aligned} \tag{39}$$

uniformly in $0 < \theta \leqslant \pi - \epsilon$, $0 < \epsilon < \pi$, where c is a fixed positive number.

If $\alpha = \beta = 0$ so that $\mu = \frac{1}{2}$, then we can replace asymptotic equality by equality in (37) and the series is uniformly convergent for $0 \leqslant \theta \leqslant \theta_0 - \epsilon$, $0 < \epsilon < \theta_0$, where $\theta_0 = 2(2^{1/2} - 1)\pi$, and

$$P_n(\cos\theta) = \left(\frac{\theta}{\sin\theta}\right)^{1/2} \sum_{m=0}^{p-1} \frac{(-)^m f_m(\theta)\, \theta^m J_m[(n+\frac{1}{2})\,\theta]}{(n+\frac{1}{2})^m} + O(n^{-p-1/2}). \tag{40}$$

Here $f_m(\theta)$ is given by (38) with $\mu = \frac{1}{2}$. Equation (40) has also been proved by Szegö (1932).

A number of inequalities are known for the Jacobi polynomials. In view of our applications it is sufficient to quote the following [see Szegö (1959)]. We take $\alpha > -1$, $\beta > -1$.

Then

$$\max_{-1\leqslant x\leqslant 1} |P_n^{(\alpha,\beta)}(x)| = \max |P_n^{(\alpha,\beta)}(\pm 1)| = \frac{(q+1)_n}{n!} = \frac{n^q}{\Gamma(q+1)}[1+O(n^{-1})],$$

$$q = \max(\alpha,\beta) \geqslant -\tfrac{1}{2}, \tag{41}$$

$$\max_{-1\leqslant x\leqslant 1} |P_n^{(\alpha,\beta)}(x)| = |P_n^{(\alpha,\beta)}(x^*)| = n^{-1/2}[1+O(n^{-1})],$$

$$q = \max(\alpha,\beta) < -\tfrac{1}{2}, \tag{42}$$

where x^* is one of the two maximum points nearest $x_0 = (\beta-\alpha)/\lambda$. Thus,

$$P_n^{(\alpha,\beta)}(x) = O(n^q), \qquad -1 \leqslant x \leqslant 1, \quad n\to\infty, \quad q = \max(-\tfrac{1}{2},\alpha,\beta) \tag{43}$$

We also have

$$\frac{d^m P_n^{(\alpha,\beta)}(x)}{dx^m} = O(n^\sigma), \qquad -1 \leqslant x \leqslant 1, \qquad n\to\infty$$

$$\sigma = \max(2m+\alpha, 2m+\beta, m-\tfrac{1}{2}) \tag{44}$$

Better estimates can be deduced from 7.2(8) or 7.4.2(8) and 7.4.3(23). See also (33)–(40).

In many applications, it is more convenient to use the shifted Jacobi polynomial defined by 8.1(26). We have the connecting relations

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1) \qquad P_n^{(\alpha,\beta)}(x) = R_n^{(\alpha,\beta)}[(1+x)/2], \tag{45}$$

so that expressions for $R_n^{(\alpha,\beta)}(x)$ corresponding to those for $P_n^{(\alpha,\beta)}(x)$ are easily set down. It is convenient to record some of these for the applications. We also develop some results for repeated integrals of the shifted polynomials.

$$R_n^{(\alpha,\beta)}(x) = [(\alpha+1)_n/n!]\,{}_2F_1\left(\begin{matrix}-n, n+\lambda\\ \alpha+1\end{matrix}\middle|\, 1-x\right)$$

$$= (-)^n[(\beta+1)_n/n!]\,{}_2F_1\left(\begin{matrix}-n, n+\lambda\\ \beta+1\end{matrix}\middle|\, x\right) \tag{46}$$

$$R_n^{(\alpha,\beta)}(1-x) = (-)^n R_n^{(\beta,\alpha)}(x) \tag{47}$$

$$R_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!, \qquad R_n^{(\alpha,\beta)}(0) = (-)^n(\beta+1)_n/n! \tag{48}$$

$$n!\,R_n^{(\alpha,\beta)}(x) = (-)^n(1-x)^{-\alpha}x^{-\beta}(d^n/dx^n)\{(1-x)^{\alpha+n}x^{\beta+n}\} \tag{49}$$

In some considerations, we need the repeated integrals of

$$Q_n^{(\alpha,\beta,\nu)}(x) \equiv Q_{n,0}^{(\alpha,\beta,\nu)}(x) = x^\nu R_n^{(\alpha,\beta)}(x), \tag{50}$$

which we define by

$$\begin{aligned} Q_{n,r}^{(\alpha,\beta,\nu)}(x) &= \int_0^x Q_{n,r-1}^{(\alpha,\beta,\nu)}(t)\,dt \\ &= \frac{1}{(r-1)!}\int_0^x (x-t)^{r-1}t^\nu R_n^{(\alpha,\beta)}(t)\,dt, \qquad R(\nu) > -1. \end{aligned} \tag{51}$$

If $\nu = 0$, we simply write

$$Q_{n,r}^{(\alpha,\beta,0)}(x) = R_{n,r}^{(\alpha,\beta)}(x); \qquad R_{n,0}^{(\alpha,\beta)}(x) \equiv R_n^{(\alpha,\beta)}(x). \tag{52}$$

Clearly,

$$Q_{n,r}^{(\alpha,\beta,\nu)}(x) = \frac{(-)^n(\beta+1)_n x^{\nu+r}}{n!\,(\nu+1)_r}\,{}_3F_2\left(\begin{matrix}-n, n+\lambda, \nu+1\\ \beta+1, \nu+r+1\end{matrix}\middle|\, x\right), \tag{53}$$

$$Q_{n,r}^{(\alpha,\nu,\nu)}(x) = \frac{(\nu+1)_n x^{\nu+r}}{(\nu+1)_r(\nu+r+1)_n} R_n^{(\alpha-r,\nu+r)}(x), \tag{54}$$

$$Q_{n,r}^{(\alpha,\nu,\nu)}(1) = \frac{(-)^n(\nu+1)_n}{n!\,(\nu+1)_r(\nu+r+1)_n(r-\alpha)_{-n}}, \tag{55}$$

$$Q_{n,r}^{(\alpha,\nu,\nu)}(1) = 0 \tag{56}$$

for α a positive integer or zero and $r - \alpha = 1, 2, \ldots, n$,

$$Q_{n,n}^{(0,\nu,\nu)}(x) = \frac{(-)^n x^{n+\nu}(1-x)^n}{n!}, \tag{57}$$

$$|Q_n^{(\alpha,\beta,\nu)}(x)| \underset{0\leqslant x\leqslant 1}{\leqslant} \frac{x^{r+R(\nu)}}{(R(\nu)+1)_r} \max_{0\leqslant x\leqslant 1} |R_n^{(\alpha,\beta)}(x)|. \tag{58}$$

Estimates of $Q_{n,r}^{(\alpha,\beta,\nu)}(x)$ for large n with α, β, ν, and r fixed and $0 < x \leqslant 1$ are readily deduced from 7.4.2(8) and 7.4.3(23). Thus,

$$\begin{aligned} Q_{n,r}^{(\alpha,\beta,\nu)}(x) = {} & \frac{(-)^n(\sin\theta/2)^{2r-2}(n+\lambda)^{\beta-2\nu-2}}{(r-1)!\,\Gamma(\beta-\nu)}[1 + O((n+\lambda)^{-1})] \\ & + (-)^n \frac{(\sin\theta/2)^{r+2\nu-\beta-1/2}(\cos\theta/2)^{r-\alpha-1/2}}{(n+\lambda)^{r+1/2}\Gamma(\frac{1}{2})} \\ & \times \cos\{(n+\lambda/2)\,\theta - (\pi/2)(\beta+r+\tfrac{1}{2}) + O((n+\lambda)^{-1})\} \\ & \times [1 + O((n+\lambda)^{-1})], \qquad x = \sin^2\theta/2, \quad 0 < x < 1, \end{aligned} \tag{59}$$

$$Q_{n,r}^{(\alpha,\beta,\nu)}(1) = \frac{(-)^n\Gamma(\nu+1)\,N^{\beta-2\nu-2}}{(r-1)!\,\Gamma(\beta-\nu)}\left[1-\frac{\alpha\beta}{2N}+O(N^{-2})\right]$$
$$+\frac{N^{\alpha-2r}}{\Gamma(\alpha+1-r)}\left[1-\frac{\alpha\beta}{2N}+O(N^{-2})\right],$$
$$N^2 = n(n+\lambda) \tag{60}$$

An alternate development of (60) can be proved as follows From 3 13 3(11),

$$Q_{n,r}^{(\alpha,\beta,\nu)}(1) = A + B, \tag{61}$$

$$A = \frac{(-)^n\Gamma(n+\beta+1)\,\Gamma(\nu+1-n-\lambda)}{\Gamma(-n-\alpha)\,\Gamma(\nu+r+1-n-\lambda)\,\Gamma(2n+\lambda+1)}$$
$$\times\,{}_3F_2\left(\begin{matrix} n+\lambda,\, n+\alpha+1,\, n+\lambda-\nu-r \\ n+\lambda-\nu,\, 2n+\lambda+1 \end{matrix}\middle|\,1\right), \tag{62}$$

$$B = \frac{(-)^n\Gamma(\nu+1)\,\Gamma(n+\beta+1)\,\Gamma(n+\lambda-1-\nu)}{\Gamma(\beta-\nu)(r-1)!\,\Gamma(n+\nu+2)\,\Gamma(n+\lambda)}$$
$$\times\,{}_3F_2\left(\begin{matrix} \nu+1 & \nu+1-\beta,\, 1-r \\ \nu+2-n-\lambda & n+\nu+2 \end{matrix}\middle|\,1\right) \tag{63}$$

For A, use 3 13 3(14, 16) in the form

$$\begin{aligned} F_p(0,4,5) = F_p(0,2,3) \qquad & a = n+\lambda, \qquad & b = n+\alpha+1, \\ c = n+\lambda-r, \qquad & e = n+\lambda-\nu, \qquad & f = 2n+\lambda+1 \end{aligned} \tag{64}$$

Thus,

$$A = \frac{\Gamma(n+\beta+1)\,\Gamma(n+\alpha+1)\,\Gamma(n+\lambda-\nu-r)}{\Gamma(n+\beta+1-\nu)\,\Gamma(\alpha+1-r)(n+r)!\,\Gamma(n+\lambda)}$$
$$\times\,{}_3F_2\left(\begin{matrix} -\nu,\, n+1 & r-\alpha \\ n+r+1 & n+\beta+1-\nu \end{matrix}\middle|\,1\right)$$

and from this we deduce

$$A = \frac{N^{\alpha-2r}}{\Gamma(\alpha+1-r)}\left[1-\frac{\alpha\beta}{2N}+O(N^{-2})\right] \tag{65}$$

Now use 3 13 3(31) for B with

$$\begin{aligned} &g = 2, && F_p(0) = F_p(0,4,5), && F_p(1) = F_p(1,4,5), \\ &a = \nu+1-\beta, && b = \nu+1 && c = 1-r, \\ &e = \nu+2-n-\lambda, && && \quad = n+\nu+2 \end{aligned} \tag{66}$$

Then

$$B = \frac{(-)^n \Gamma(n+\beta+1)\,\Gamma(n+\lambda-r-\nu)(n+r-1)!\,\Gamma(\nu+1)}{\Gamma(n+\lambda+1-r)\,\Gamma(n+r+\nu+1)n!\,(r-1)!\,\Gamma(\beta-\nu)}$$
$$\times {}_3F_2\left(\begin{matrix}1-r, \alpha+1-r, \nu+1\\ 1-r-n, n+\lambda+1-r\end{matrix}\middle|\, 1\right), \tag{67}$$

where $B = 0$ if $r = 0$ and for $r \geqslant 1$, the latter ${}_3F_2$ is to be interpreted as a finite sum of $(r-1)$ terms. We find that

$$B = \frac{(-)^n \Gamma(\nu+1)\, N^{\beta-2\nu-2}}{(r-1)!\,\Gamma(\beta-\nu)}\left[1 - \frac{\alpha\beta}{2N} + O(N^{-2})\right], \tag{68}$$

and the combination (61), (65), (68) yields (60).

8.3. Expansion of Functions in Series of Jacobi Polynomials

Suppose formally that

$$f(x) = \sum_{n=0}^{\infty} c_n P_n^{(\alpha,\beta)}(x). \tag{1}$$

Multiply both sides by $(1-x)^\alpha(1+x)^\beta P_m^{(\alpha,\beta)}(x)$ and integrate from -1 to 1. Then in view of 8.2(22),

$$c_n = (h_n)^{-1}\int_{-1}^{1} f(x)(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)\,dx. \tag{2}$$

The coefficients c_n are often called the Fourier coefficients associated with $f(x)$. The evaluation and estimation of these quantities for various types of functions are discussed in 8.4. Here we are concerned with the representation of $f(x)$ by (1). As in the case of Fourier series, we seek conditions to insure that the series converges and that its sum is $f(x)$.

We quote the following theorem.

Theorem 1. *Let $f(x)$ be Lebesgue-measurable in $-1 \leqslant x \leqslant 1$, and let the integrals*

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\,|f(x)|\,dx, \qquad \int_{-1}^{1}(1-x)^{\alpha/2-1/4}(1+x)^{\beta/2-1/4}\,|f(x)|\,dx$$

exist. Let $s_n(x)$ denote the nth partial sum of the expansion of $f(x)$ in series of Jacobi polynomials, and $s_n^*(x)$ the nth partial sum of the Fourier cosine series of

$$(1-\cos\theta)^{\alpha/2+1/4}(1+\cos\theta)^{\beta/2+1/4} f(\cos\theta).$$

Then for $-1 < x < 1$,

$$\lim_{n\to\infty} \{s_n(x) - (1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4} s_n^*(x)\} = 0,$$

uniformly in $-1+\epsilon \leqslant x \leqslant 1-\epsilon$, $0 < \epsilon < 1$

This is called an equiconvergence theorem For proof, see Szegö (1959, p 244) Clearly, application of the theorem requires knowledge of the convergence of the Fourier series for $f(x)$ In this connection, it is sufficient to quote the following result [see Zygmund (1959)]

Theorem 2 *Let $f(x)$ be periodic and have period 2π If $f'(x)$ is continuous in $-\pi \leqslant x \leqslant \pi$ except for a finite number of bounded jumps, then the Fourier series for $f(x)$ converges pointwise to $\frac{1}{2}[f(x-0)+f(x+0)]$ Furthermore, the convergence is uniform in any closed interval which does not include a point of discontinuity of $f(x)$*

Rau (1950) has proved the following theorem

Theorem 3. *If $f(x)$ is continuous in the closed interval $-1 \leqslant x \leqslant 1$ and has a piecewise continuous derivative there, then with $\alpha > -1$, $\beta > -1$, the Jacobi series* (1) *associated with $f(x)$ converges uniformly to $f(x)$ in $-1+\epsilon \leqslant x \leqslant 1-\epsilon$, $0 < \epsilon < 1$*

In the case of analytic functions [see Szego (1959, p 243)] we have

Theorem 4. *If $f(x)$ is analytic in the closed interval $-1 \leqslant x \leqslant 1$, then the Jacobi series* (1) *associated with $f(x)$ is convergent in the interior of the largest ellipse with foci at ± 1 in which $f(x)$ is analytic*

For an application of Theorem 1, we have

$$x^\mu = \Gamma(\mu+\beta+1) \sum_{n=0}^{\infty} \frac{(-)^n(2n+\lambda)\,\Gamma(n+\lambda)(-\mu)_n}{\Gamma(n+\beta+1)\,\Gamma(n+\lambda+\mu+1)} R_n^{(\alpha,\beta)}(x) \tag{3}$$

valid for

$$\begin{gathered} \alpha > -1, \quad \beta > -1 \quad -R(\mu) < \min(\beta+1, \beta/2+3/4), \\ \epsilon \leqslant x \leqslant 1-\epsilon, \quad 0 < \epsilon < 1 \end{gathered} \tag{4}$$

The coefficient of $R_n^{(\alpha,\beta)}(x)$ in (3) readily follows from 8 2(27, 30) Suppose $R(\mu) > 0$ If $x \to 0$, $x^\mu \to 0$, the series on the right-hand side of (3) converges and its sum is zero Since $\beta > -1$, $\min(0, \beta+1, \beta/2+3/4) = 0$ and so (3) is valid for

$$\alpha > -1, \quad \beta > -1, \quad R(\mu) > 0, \quad 0 \leqslant x \leqslant 1-\epsilon, \quad 0 < \epsilon < 1 \tag{5}$$

If $x = 1$, the right-hand side of (3) converges to unity if $R(\mu) > \frac{1}{2}(\alpha - \beta)$. Hence (3) is valid for

$$\alpha > -1, \qquad \beta > -1, \qquad -R(\mu) < \min(0, \tfrac{1}{2}(\beta - \alpha)), \qquad 0 \leqslant x \leqslant 1. \tag{6}$$

With the aid of 8.2(43) and 2.11(11), it readily follows that (3) holds in $0 \leqslant x \leqslant 1$ when $R(\mu) > \frac{1}{2}(q - \beta)$, that is, when

$$-R(\mu) < \min(0, \tfrac{1}{2}(\beta - \alpha), \tfrac{1}{2}\beta + \tfrac{1}{4}).$$

However, the latter is more restrictive than (6). We can also show that the equiconvergence theorem gives better conditions for (3) than Rau's theorem.

If μ is a positive integer or zero, x^μ is analytic, and (3) is valid for all x.

An easily proved generalization of (3) is

$$x^\mu e^{zx} = \Gamma(\mu + \beta + 1) \sum_{n=0}^{\infty} \frac{(-)^n(2n + \lambda)\,\Gamma(n + \lambda)(-\mu)_n}{\Gamma(n + \beta + 1)\,\Gamma(n + \lambda + \mu + 1)}$$
$$\times {}_2F_2\left(\begin{matrix} \mu + 1, \mu + \beta + 1 \\ \mu - n + 1, \mu + n + \lambda + 1 \end{matrix} \middle| z\right) R_n^{(\alpha,\beta)}(x), \tag{7}$$

valid under any of the conditions (4)–(6) and $|z| < \infty$. For further generalizations of (7), see 9.1(1, 12) and 9.2(2, 7, 14).

Another expansion of interest is

$$x^\rho = \sum_{k=0}^{\infty} c_{k,\rho} P_k^{(\alpha,\beta)}(x),$$
$$\alpha > -1, \qquad \beta > -1, \qquad R(\rho) \geqslant 0, \qquad 0 < x < 1, \tag{8}$$

where

$$c_{k,\rho} = \frac{(-)^k(-\rho)_k 2^\rho(2k + \lambda)\,\Gamma(\beta + \rho + 1)\,\Gamma(k + \lambda)}{\Gamma(k + \beta + 1)\,\Gamma(k + \rho + \lambda + 1)}\, {}_2F_1\left(\begin{matrix} k - \rho, -k - \rho - \lambda \\ -\rho - \beta \end{matrix} \middle| \frac{1}{2}\right)$$
$$+ \frac{(-)^{k+1}\pi(2k + \lambda)\,\Gamma(k + \lambda)\, e^{i\pi(\rho+\beta)}}{2^{\beta+1}\Gamma(k + \alpha + 1)\,\Gamma(\rho + \beta + 2)\,\Gamma(-\rho)\cos\pi(\rho + \beta)}$$
$$\times {}_2F_1\left(\begin{matrix} -k - \alpha, k + \beta + 1 \\ +\beta + 2 \end{matrix} \middle| \frac{1}{2}\right). \tag{9}$$

If $\alpha = \beta$,

$$c_{k,\rho} = \frac{(-\rho)_k(2k + 2\alpha + 1)\,\Gamma(k + 2\alpha + 1)\,\Gamma[(1 + \rho - k)/2]}{2^{k+2\alpha+2}\Gamma(k + \alpha + 1)\,\Gamma[(k + 2\alpha + \rho + 3)/2]}$$
$$\times [(-)^k + e^{i\pi\rho}]. \tag{10}$$

If, further, ρ is a positive integer or zero, we can recover 8.2(29).

8 4. Evaluation and Estimation of the Coefficients in the Expansion of a Given Function $f(x)$ in Series of Jacobi Polynomials

8 4 1 The Coefficients as an Integral Transform

We suppose throughout that $\alpha > -1$, $\beta > -1$ From 8 3(1, 2), we have formally at least

$$f(x) = \sum_{n=0}^{\infty} c_n P_n^{(\alpha,\beta)}(x) \tag{1}$$

$$c_n = (h_n)^{-1} \int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)\, dx \tag{2}$$

When $\alpha = \beta$, $\alpha = -\frac{1}{2}$, an analysis of approximations for c_n, which follow from (2) by application of two trapezoidal type rules of integration, is given in 8 5 4 In particular, see the remarks following 8 5 4(11)

From (2) and 8 2(7) we get

$$c_n = \frac{(-)^n (h_n)^{-1}}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} \{(1-x)^{\alpha+n}(1+x)^{\beta+n}\}\, dx, \tag{3}$$

and after integrating (3) by parts n times, it follows that

$$c_n = \frac{(-)^n (h_n)^{-1}}{2^n n!} \int_{-1}^{1} f^{(n)}(x)(1-x)^{\alpha+n}(1+x)^{\beta+n}\, dx \tag{4}$$

We note that (4) is a beta transform Thus if $f(x)$ is a member of the hypergeometric family, so also is c_n as readily follows from (4), 3 4 (1), and 3 6(10) Likewise, if $f(x)$ is a G function, then c_n is also a G-function in view of 5 6 4(10) Other types of transforms are useful to identify c_n In this connection the references noted at the end of 3 2 are pertinent

For a general illustration, suppose

$$f(x) = \int_{c}^{d} h(x\ t)\, g(t)\, dt \tag{5}$$

$$h(x\ t) = \sum_{n=0}^{\infty} b_n(t)\, P_n^{(\alpha,\beta)}(x) \tag{6}$$

Combine (5) and (6), interchange the order of integration and summation (which we assume is valid), and compare with (1) to get

$$c_n = \int_{c}^{d} b_n(t)\, g(t)\, dt \tag{7}$$

Observe that both $f(x)$ and c_n are from the same family of transforms. Thus if $f(x)$ is defined by (5) and the kernel of the transform has a known expansion in series of Jacobi polynomials, then c_n is defined by a like transform with kernel $b_n(t)$.

For a more concrete but still general example, suppose $f(x)$ is the Laplace transform of a known function $g(t)$. Thus,

$$f(x) = \int_0^\infty e^{-xt} g(t)\, dt. \tag{8}$$

Now it can be shown that [see 9.3.2(2) and 9.3.3(7)]

$$e^{-xt} = \sum_{n=0}^{\infty} \frac{(-)^n \Omega_n I_{n+\alpha+1/2}(t)}{t^{\alpha+1/2}} P_n^{(\alpha,\alpha)}(x), \qquad -1 \leqslant x \leqslant 1,$$

$$\Omega_n = \frac{(2\pi)^{1/2}(n+\alpha+\frac{1}{2})\,\Gamma(n+2\alpha+1)}{2^\alpha \Gamma(n+\alpha+1)}, \qquad \alpha \neq -\tfrac{1}{2},$$

$$\Omega_n = n!\,\epsilon_n/(\tfrac{1}{2})_n, \quad \epsilon_0 = 1, \quad \epsilon_n = 2 \quad \text{if} \quad n > 0, \quad \text{when} \quad \alpha = -\tfrac{1}{2}. \tag{9}$$

Thus,

$$c_n = (-)^n \Omega_n \int_0^\infty t^{-\alpha-1/2} I_{n+\alpha+1/2}(t)\, g(t)\, dt, \tag{10}$$

which is a Hankel transform, see Erdélyi *et al.* (1954). We write

$$\mathscr{H}\{F(t), y, \nu\} = \int_0^\infty F(t)\, J_\nu(yt)(yt)^{1/2}\, dt. \tag{11}$$

Then

$$c_n = \Omega_n \exp[i\pi(n-\alpha-1)/2]\, \mathscr{H}\{g(t)/t^{\alpha+1}, e^{i\pi/2}, n+\alpha+\tfrac{1}{2}\}. \tag{12}$$

The case $\alpha = -\frac{1}{2}$ is important for the applications. Thus with

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x), \tag{13}$$

and $f(x)$ also given by (8), then

$$a_n = \epsilon_n \exp[i\pi(n-\tfrac{1}{2})/2]\, \mathscr{H}\{g(t)/t^{1/2}, e^{i\pi/2}, n\}. \tag{14}$$

For expansions in series of the "shifted" polynomials, suppose

$$f(x) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\alpha)}(x), \tag{15}$$

$$f(x) = \int_0^\infty e^{-xt} g(t)\, dt. \tag{16}$$

Since

$$e^{-xt} = \sum_{n=0}^{\infty} \frac{(-)^n \Omega_n e^{-t/2} I_{n+\alpha+1/2}(t/2)}{(t/2)^{\alpha+1/2}} R_n^{(\alpha,\alpha)}(x), \tag{17}$$

we find

$$\begin{aligned} a_n &= 2(-)^n \Omega_n \int_0^\infty \frac{e^{-t} I_{n+\alpha+1/2}(t)\, g(2t)}{t^{\alpha+1/2}}\, dt, \\ a_n &= 2\Omega_n \exp\{i\pi(n-\alpha-1)/2\}\, \mathscr{H}\{[e^{-t}g(2t)]/t^{\alpha+1}, e^{i\pi/2}, n+\alpha+\tfrac{1}{2}\} \end{aligned} \tag{18}$$

In particular, with (16) and

$$f(x) = \sum_{n=0}^{\infty} c_n T_n^*(x), \tag{19}$$

$$c_n = 2\epsilon_n \exp\{i\pi(n-\tfrac{1}{2})/2\}\, \mathscr{H}\{[e^{-t}g(2t)]/t^{1/2}, e^{i\pi/2}, n\} \tag{20}$$

The formulations (8)–(20) are due to Wimp (1961) For an application of these results to $\ln \Gamma(z)$ and its derivatives, see 2 10 2

Next suppose that $f(x)$ is defined as a Fourier transform Let

$$f(x) = \int_0^\infty e^{ixt} g(t)\, dt = f_1(x) + if_2(x), \tag{21}$$

$$f_1(x) = \int_0^\infty (\cos xt)\, g(t)\, dt, \qquad f_2(x) = \int_0^\infty (\sin xt)\, g(t)\, dt, \tag{22}$$

$$f_1(x) = \sum_{n=0}^{\infty} C_n P_n^{(\alpha,\alpha)}(x), \qquad f_2(x) = \sum_{n=0}^{\infty} S_n P_n^{(\alpha,\alpha)}(x) \tag{23}$$

Then

$$C_{2n+1} = 0, \qquad C_{2n} = (-)^n \Omega_{2n} \mathscr{H}\{g(t)\, t^{\alpha+1}, 1, 2n+\alpha+\tfrac{1}{2}\}, \tag{24}$$

$$S_{2n} = 0, \qquad S_{2n+1} = (-)^n \Omega_{2n+1} \mathscr{H}\{g(t)\, t^{\alpha+1}, 1, 2n+\alpha+\tfrac{3}{2}\} \tag{25}$$

In particular, using (21), (22), $\alpha = -\frac{1}{2}$, and

$$f_1(x) = \sum_{n=0}^{\infty} c_n T_{2n}(x) \qquad f_2(x) = \sum_{n=0}^{\infty} s_n T_{2n+1}(x), \tag{26}$$

we have

$$c_n = (-)^n \epsilon_n \mathscr{H}\{g(t)/t^{1/2}, 1, 2n\}, \qquad s_n = 2(-)^n \mathscr{H}\{g(t)/t^{1/2}, 1, 2n+1\} \tag{27}$$

We next consider the situation where $f(x)$ is defined by an inverse Laplace transform. This approach is due to Elliott and Szekeres (1965). Suppose

$$f(x) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\alpha)}(x), \tag{28}$$

$$g(p) = \int_0^{\infty} e^{-px} f(x)\, dx, \tag{29}$$

or

$$f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} e^{xz} g(z)\, dz, \tag{30}$$

where $c > 0$ and c lies to the right of all singularities of $g(z)$. Then

$$a_n = (2\pi i)^{-1} \Omega_n \int_{c-i\infty}^{c+i\infty} \frac{e^{z/2} I_{n+\alpha+1/2}(z/2)\, g(z)}{(z/2)^{\alpha+1/2}}\, dz. \tag{31}$$

Let

$$z = c - \rho e^{i\theta}, \qquad -\pi/2 \leqslant \theta \leqslant \pi/2$$

and consider the locus of points $L_c = \{z \mid R(z) \leqslant c\}$. If

$$\lim_{z\to\infty} z^{1/2} g(z) = 0, \qquad z \text{ on } L_c,$$

then

$$a_n = \text{sum of residues of } \frac{\Omega_n e^{z/2} I_{n+\alpha+1/2}(z/2)\, g(z)}{(z/2)^{\alpha+1/2}}. \tag{32}$$

Suppose now that

$$f(x) = \sum_{n=0}^{\infty} c_n P_n^{(\alpha,\alpha)}(x). \tag{33}$$

Then from (30), $f(x) = 0$ for $x < 0$ and the right-hand side of (30) gives $\frac{1}{2}f(0)$ for $x = 0$. To account for negative values of x, let us define

$$\left.\begin{aligned} f_1(x) &= f(x), & x &> 0 \\ &= \tfrac{1}{2}f(0), & x &= 0 \\ &= 0, & x &< 0 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} f_2(x) &= f(x), & x &< 0 \\ &= \tfrac{1}{2}f(0), & x &= 0 \\ &= 0, & x &> 0 \end{aligned}\right\}, \tag{34}$$

so that for all real x,

$$f(x) = f_1(x) + f_2(x). \tag{35}$$

Let

$$g_1(p) = \int_0^\infty e^{-px} f(x)\,dx, \qquad g_2(p) = \int_0^\infty e^{-px} f(-x)\,dx, \tag{36}$$

$$f_1(x) = (2\pi i)^{-1} \int_{c_1-i\infty}^{c_1+i\infty} e^{xz} g_1(z)\,dz, \tag{37}$$

$$\begin{aligned} f_2(-x) &= (2\pi i)^{-1} \int_{c_2-i\infty}^{c_2+i\infty} e^{xz} g_2(z)\,dz, \qquad x > 0, \\ f_2(x) &= (2\pi i)^{-1} \int_{-c_2-i\infty}^{-c_2+i\infty} e^{xz} g_2(-z)\,dz, \qquad x < 0 \end{aligned} \tag{38}$$

Here $c_r > 0$ and c_r lies to the right of all singularities of $g_r(z)$, $r = 1, 2$ Thus

$$f(x) = (2\pi i)^{-1} \left\{ \int_{c_1-i\infty}^{c_1+i\infty} e^{xz} g_1(z)\,dz + \int_{-c_2-i\infty}^{-c_2+i\infty} e^{xz} g_2(-z)\,dz \right\} \tag{39}$$

and so

$$c_n = (2\pi i)^{-1} Q_n \left\{ \int_{c_1-i\infty}^{c_1+i\infty} \frac{I_{n+\alpha+1/2}(z)\, g_1(z)}{z^{\alpha+1/2}}\,dz + \int_{-c_2-i\infty}^{-c_2+i\infty} \frac{I_{n+\alpha+1/2}(z)\, g_2(-z)}{z^{\alpha+1/2}}\,dz \right\} \tag{40}$$

Again we remark that if $f(z)$ can be expressed as a G-function, then the same is true for c_n In this instance see the material in 5 6

8 4 2 Evaluation of the Coefficients when $f(x)$ Is Defined by a Taylor Series

It is convenient to generalize and present work due to Fields and Wimp (1963) Let $f(x)$ be analytic at $x = 0$ and write

$$f(\omega x) = \sum_{k=0}^{\infty} \xi_k \omega^k x^k, \qquad \xi_k = f^{(k)}(0)/k! \tag{1}$$

Let $g_n(x)$ be a polynomial in x of degree n Then there exist constants $\sigma_{k,n}$ such that

$$x^k = \sum_{n=0}^{k} \sigma_{k,n} g_n(x) \tag{2}$$

Combining (1) and (2), we have

$$f(\omega x) = \sum_{n=0}^{\infty} C_n g_n(x), \tag{3}$$

$$C_n = \sum_{k=n}^{\infty} \sigma_{k,n} \xi_k \omega^k. \tag{4}$$

Note that if $\sigma_{k,n} = O(1)$ uniformly in n as $k \to \infty$, then most certainly (4) converges if $|\omega|$ is less than the radius of convergence of $f(x)$. Equation (3) is called the basic series for $f(\omega x)$ corresponding to the set of functions $\{g_n(x)\}$. For a general discussion of basic series, see the work of Boas and Buck (1958). Here we consider the basic series where $g_n(x)$ is either the extended Jacobi polynomial

$$G_n(x, \lambda) = {}_{p+2}F_q\left(\begin{matrix} -n, n+\lambda, \alpha_p \\ \rho_q \end{matrix} \middle| x\right) \tag{5}$$

or its confluent form, the extended Laguerre polynomial

$$G_n(x) = \lim_{\lambda\to\infty} G_n(x/\lambda, \lambda) = {}_{p+1}F_q\left(\begin{matrix} -n, \alpha_p \\ \rho_q \end{matrix} \middle| x\right). \tag{6}$$

In the sequel, we use the notation

$$x^k = \sum_{n=0}^{k} \sigma_{k,n}(\lambda)\, G_n(x, \lambda),$$

$$f(\omega x) = \sum_{n=0}^{\infty} C_n(\omega, \lambda)\, G_n(x, \lambda),$$

$$C_n(\omega, \lambda) = \sum_{k=n}^{\infty} \sigma_{k,n}(\lambda)\, \xi_k \omega^k, \qquad \xi_k = f^{(k)}(0)/k!\,, \tag{7}$$

and if $G_n(x, \lambda)$ is replaced by $G_n(x)$, we simply replace $\sigma_{k,n}(\lambda)$ and $C_n(\omega, \lambda)$ by $\sigma_{k,n}$ and $C_n(\omega)$, respectively.

The principal result of Fields and Wimp is as follows. Let none of the quantities $\lambda + 1$, α_j, and ρ_j be a negative integer or zero. Then

$$\sigma_{k,n}(\lambda) = \frac{(-k)_n (2n+\lambda)(\rho_q)_k}{n!\,(n+\lambda)(n+\lambda+1)_k(\alpha_p)_k}, \tag{8}$$

$$C_n(\omega, \lambda) = \frac{(-)^n \omega^n (\rho_q)_n}{(n+\lambda)_n(\alpha_p)_n} \sum_{k=0}^{\infty} \frac{(n+1)_k (n+\rho_q)_k \xi_{k+n} \omega^k}{(2n+\lambda+1)_k (n+\alpha_p)_k k!}, \tag{9}$$

and for the confluent case,

$$\sigma_{k,n} = \frac{(-k)_n(\rho_q)_k}{n!\,(\alpha_p)_k}, \tag{10}$$

$$C_n(\omega) = \frac{(-)^n\omega^n(\rho_q)_n}{(\alpha_p)_n}\sum_{k=0}^{\infty}\frac{(n+1)_k(n+\rho_q)_k\xi_{k+n}\omega^k}{(n+\alpha_p)_k k!} \tag{11}$$

Note that in view of (8),

$$x^k = \frac{(\rho_q)_k}{(\alpha_p)_k}\sum_{n=0}^{k}\frac{(-k)_n(2n+\lambda)}{(n+\lambda+1)_k(n+\lambda)\,n!}\,{}_{p+2}F_q\left(\begin{matrix}-n,\,n+\lambda\;\;\alpha_p\\ \rho_q\end{matrix}\middle|\,x\right) \tag{12}$$

The proof is by induction on p and q using Laplace transform techniques When $p = 0$, $q = 1$, $\rho_1 = \beta + 1$, and $\lambda = \alpha + \beta + 1$, (12) may be deduced from the classical result 8 2(30) Thus (12) is also true when $p = q = 0$ by confluence Suppose then that (12) is true for some p and q In (12) replace x by xt, multiply both sides of this equation by $e^{-t}t^{\tau-1}$ and apply 3 6(13) With an obvious change of notation, we get (12) with p replaced by $p + 1$ and the induction on p is complete In the above derivation we must suppose $R(\tau) > 0$, but this condition can be relaxed by continuity For the induction on q, in (12) replace x by x/t, multiply both sides of this equation by $e^{t}t^{\tau}$ and take the inverse Laplace transform, that is, use 3 6(19)

For the expansion of $f(z)$ in series of the shifted Jacobi polynomials, we have

$$f(\omega x) = \sum_{n=0}^{\infty} S_n(\omega)\,R_n^{(\alpha,\beta)}(x), \tag{13}$$

$$S_n(\omega) = \frac{n!\,\omega^n}{(n+\lambda)_n}\sum_{k=0}^{\infty}\frac{(n+\beta+1)_k(n+1)_k\xi_{k+n}\omega^k}{(2n+\lambda+1)_k k!} \tag{14}$$

When $\alpha = \beta = -\frac{1}{2}$, that is, $\lambda = 0$, we get the important expansion in series of shifted Chebyshev polynomials of the first kind Thus,

$$f(\omega x) = \sum_{n=0}^{\infty} A_n(\omega)\,T_n^*(x), \qquad A_0(\omega) = \sum_{k=0}^{\infty}[(\tfrac{1}{2})_k\xi_k\omega^k]/k!\,,$$

$$A_n(\omega) = 2\left(\frac{\omega}{4}\right)^n\sum_{k=0}^{\infty}\frac{(n+\frac{1}{2})_k(n+1)_k\xi_{k+n}\omega^k}{(2n+1)_k k!} \qquad n \geqslant 1 \tag{15}$$

In a similar fashion, using 8 2(28), we can derive an expression for

$f(\omega x)$ in series of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. The case $\alpha = \beta$ is of particular interest. Thus from 8.2(29) and (3), (4), we find

$$x^\epsilon f(\omega x^2) = \sum_{n=0}^{\infty} E_n(\omega)\, P_{2n+\epsilon}^{(\alpha,\alpha)}(x), \qquad \epsilon = 0 \quad \text{or} \quad \epsilon = 1,$$

$$E_n(\omega) = \frac{2^\epsilon(2n+\epsilon)!\,(4\omega)^n}{(2n+2\alpha+1+\epsilon)_{2n+\epsilon}} \sum_{k=0}^{\infty} \frac{(n+\frac{1}{2}+\epsilon)_k(n+1)_k\xi_{k+n}\omega^k}{(2n+\alpha+\frac{3}{2}+\epsilon)_k k!}, \tag{16}$$

where ξ_k has the same meaning as in (1). For the case $\alpha = -\frac{1}{2}$, we have

$$x^\epsilon f(\omega x^2) = \sum_{n=0}^{\infty} F_n(\omega)\, T_{2n+\epsilon}(x), \qquad \epsilon = 0 \quad \text{or} \quad \epsilon = 1,$$

$$F_n(\omega) = 2^{1-\epsilon}\left(\frac{\omega}{4}\right)^n \sum_{k=0}^{\infty} \frac{(n+\frac{1}{2}+\epsilon)_k(n+1)_k\xi_{k+n}\omega^k}{(2n+1+\epsilon)_k k!}. \tag{17}$$

Note that if $\epsilon = 0$, (17) and (15) coalesce when in the latter x is replaced by x^2 in view of 8.5.2(2).

The foregoing analysis may be used to derive expansions of hypergeometric functions in series of functions of the same kind. This investigation is deferred to Chapter IX where we employ a different point of view to get some very general expansion formulas.

8.4.3. Asymptotic Estimates of the Coefficients

As so much of our work bears on Chebyshev polynomials, in the sequel we consider the coefficients in the expansion 8.4.1(1) with the Jacobi polynomial replaced by $T_n(x)$, even though results could readily be obtained for the more general situation.

Consider

$$f(x) = \tfrac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k T_k(x), \qquad c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)\,T_k(x)}{(1-x^2)^{1/2}}\,dx. \tag{1}$$

Let C be a completely closed contour such that $f(x)$ is analytic on and within C. Further let the line segment $-1 \leqslant x \leqslant 1$ lie within C. By Cauchy's formula

$$f(x) = (2\pi i)^{-1}\int_C [f(z)/(z-x)]\,dz. \tag{2}$$

So

$$c_k = \frac{1}{i\pi^2}\int_C f(z)\left\{\int_{-1}^{1}\frac{T_k(x)\,dx}{(1-x^2)^{1/2}(z-x)}\right\}dz$$

$$= \frac{1}{i\pi}\int_C \frac{f(z)\,e^{-k\varphi}}{\sinh\varphi}\,dz\,, \tag{3}$$

in view of 8 5 1(41, 42) If

$$z = u + iv = \cosh\varphi \qquad \varphi = \alpha + i\beta, \tag{4}$$

then

$$u^2/\cosh^2\alpha + v^2/\sinh^2\alpha = 1, \qquad |z + (z^2-1)^{1/2}| = |e^{\varphi}| = e^{\alpha} = \rho \tag{5}$$

Thus the latter equation generates an ellipse which we call E_ρ with foci at $z = \pm 1$ and semiaxes $\cosh\alpha$ and $\sinh\alpha$, respectively Observe that if $z = x$, $-1 \leqslant x \leqslant 1$, then $\rho = 1$ and $\alpha = 0$ Introduce the transformation

$$e^{\varphi} = \sigma \tag{6}$$

This maps the exterior of the ellipse E_ρ onto the exterior of the circle C_ρ of radius ρ with center at the origin in the σ-plane In (3), let C be this circle Then

$$c_k = \frac{1}{i\pi}\int_{C_\rho}\frac{f(\frac{1}{2}(\sigma+\sigma^{-1}))}{\sigma^{k+1}}\,d\sigma, \tag{7}$$

$$|c_k| \leqslant 2M(\rho)/\rho^k, \qquad M(\rho) = \max_{z\text{ on }C_\rho}|f(z)| \tag{8}$$

If ρ^* is that ρ which minimizes this bound, then with

$$f_n(x) = \tfrac{1}{2}c_0 + \sum_{k=1}^{n} c_k T_k(x) \qquad \epsilon_n(x) = f(x) - f_n(x),$$

$$|\epsilon_n(x)| \leqslant \frac{2M(\rho^*)}{(\rho^*)^n(\rho^*-1)} \qquad \rho^* > 1 \tag{9}$$

This proves that the expansion (1) converges in the domain in which $f(x)$ is analytic Clearly the rate of convergence depends on how large an ellipse we can have in the complex plane without encountering a singularity of $f(x)$

For another point of view, it follows from 8 5 1(38) that

$$\frac{1}{x-y} = \frac{T_n'(x)}{nT_n(x)} + \frac{T_n(y)}{(x-y)\,T_n(x)} + \frac{2}{T_n(x)}\sum_{k=1}^{n-1}\frac{T_{n-k}(x)\,T_k(y)}{n-k} \tag{10}$$

and this coupled with (2) gives

$$f(x) = \tfrac{1}{2}c_0 + \sum_{k=1}^{n-1} c_k T_k(x) + \delta_{n-1}(x),$$

$$c_k = \frac{1}{i\pi}\int_C \frac{f(z)\, T'_{n-k}(z)}{(n-k)\, T_n(z)}\, dz, \qquad \delta_{n-1}(x) = \frac{T_n(x)}{2\pi i}\int_C \frac{f(z)\, dz}{(z-x)\, T_n(z)}. \tag{11}$$

To evaluate c_k, we need the residues due to the zeros of $T_n(z)$. Thus with the notation of Lemma 1 of 8.5.4, we have

$$c_k = 2\sum_{\alpha=1}^{n} \frac{T'_{n-k}(x_\alpha) f(x_\alpha)}{(n-k)\, T'_n(x_\alpha)} = \frac{2}{n}\sum_{\alpha=1}^{n} T_k(x_\alpha) f(x_\alpha). \tag{12}$$

If (12) is inserted in the expansion for $f(x)$ in (11), then this representation with c_k replaced by d_k when compared with the combination of Theorems 1 and 2 of 8.5.4 gives a formulation for the remainder $\delta_{n-1}(x)$. In this connection, see also the work of Elliott (1965). If M is the maximum of $|f(z)|$ for z on C, then

$$|\delta_{n-1}(x)| \leqslant \frac{M}{2\pi}\int_C \left|\frac{dz}{(z-x)\, T_n(z)}\right|.$$

Now take z as in (4). Then $|T_n(z)|^2 \geqslant \sinh^2 n\alpha$, $|z - x| \geqslant \cosh\alpha - 1$, and the length of C does not exceed $2\pi\cosh\alpha$. Hence,

$$|\delta_{n-1}(x)| \leqslant \frac{M\cosh\alpha}{\sinh n\alpha(\cosh\alpha - 1)}. \tag{13}$$

Notice that (9) and (13) are essentially the same when α is large if in the latter n is replaced by $n + 1$, and if in the former ρ^* is replaced by ρ.

When $f(x)$ is entire, (8) furnishes a bound for c_k. To achieve an estimate of this coefficient for large k, we follow Elliott and Szekeres (1965). The idea is to deform C in (3) such that it never passes through the branch points $z = \pm 1$, and then use the method of steepest descent. Let

$$e^{\omega(z)} = \frac{f(z)\, e^{-k\varphi}}{\sinh\varphi}, \qquad z = \cosh\varphi \tag{14}$$

so that

$$\omega'(z) = \frac{f'(z)}{f(z)} - \frac{(k + \coth\varphi)}{\sinh\varphi},$$

$$\omega''(z) = \frac{f''(z)}{f(z)} - \left\{\frac{f'(z)}{f(z)}\right\}^2 + \frac{\tfrac{1}{2}k\sinh 2\varphi + \cosh^2\varphi + 1}{\sinh^4\varphi}. \tag{15}$$

Let ξ be a number such that $\omega'(\xi) = 0$ Thus for k sufficiently large,

$$\sinh\theta[f(\xi)/f(\xi)] \sim k, \qquad \xi = \cosh\theta \tag{16}$$

Upon application of the method of steepest descent, we have for k sufficiently large

$$c_k \sim -\frac{i(2/\pi)^{1/2}\eta f(\xi)\, e^{-k\xi}}{|\omega''(\xi)|^{1/2}\sinh\xi}, \qquad \eta = \exp\{\tfrac{1}{2}i\pi - \tfrac{1}{2}i\arg\omega''(\xi)\}, \tag{17}$$

or a sum of such relations, one for each point ξ which lies on the contour

In many considerations, use of (17) is limited since solution of the transcendental equation for ξ can be a formidable task In numerous situations where (17) is applicable, the same and often better estimates can be deduced directly from known closed form expressions for the coefficients In illustration, from 9 3 2(5),

$$ {}_pF_q\left(\begin{matrix} a_p \\ b_q \end{matrix}\middle| z\omega\right) = \sum_{n=0}^{\infty} B_n T_n^*(\omega),$$

$$B_n = \epsilon_n \frac{(a_p)_n (z/4)^n}{(b_q)_n n!}\, {}_{p+1}F_{q+1}\left(\begin{matrix} \tfrac{1}{2}+n,\ a_p+n \\ 1+2n,\ b_q+n \end{matrix}\middle| z\right),$$

$$p \leqslant q \quad |z| < \infty, \quad |\omega| < \infty, \qquad p = q+1, \quad z \neq 1, \quad |\arg(1-z)| < \pi,$$
$$0 \leqslant \omega \leqslant 1 \tag{18}$$

Then for n sufficiently large, excellent estimates of B_n follow from 7 3(7–9)

For estimates of the coefficients when $f(x)$ has poles or branch points, see Elliott (1964) There it is indicated how the results can be extended to expansions in series of the Jacobi polynomials

Asymptotic estimates of the coefficients of the expansion for Whittaker functions of argument $z\omega$ in series of Chebyshev polynomials of the first kind of argument $1/\omega$ have been studied by Nemeth (1965a,b) and G F Miller (1966) These results have been generalized by Wimp (1967) [see 9 2(14–20)]

8.5. Chebyshev Polynomials

8 5 1 The Polynomials $T_n(x)$ and $U_n(x)$

The Chebyshev polynomials have already been noted in 8 1(29–32) We have

$$T_n(x) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-1/2,\,-1/2)}(x), \qquad U_n(x) = \frac{(n+1)!}{(\frac{3}{2})_n} P_n^{(1/2,\,1/2)}(x), \tag{1}$$

and it is readily shown that

$$T_n(x) = \cos n\theta, \qquad U_n(x) = \csc\theta \sin(n+1)\theta, \qquad x = \cos\theta. \tag{2}$$

Also,

$$T_{2n}(\sin\theta) = (-)^n \cos 2n\theta, \qquad T_{2n+1}(\sin\theta) = (-)^n \sin(2n+1)\theta, \tag{3}$$

$$U_{2n}(\sin\theta) = \frac{(-)^n \cos(2n+1)\theta}{\cos\theta}, \qquad U_{2n+1}(\sin\theta) = \frac{(-)^n \sin(2n+2)\theta}{\cos\theta}. \tag{4}$$

Thus many identities involving the Chebyshev polynomials are paraphrases of well-known trigonometric identities. We present a short list of results as other identities may be deduced from 8.2. Hypergeometric representations and other properties are as follows:

$$T_n(x) = {}_2F_1\left(\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}\middle|\frac{1-x}{2}\right) = (-)^n\, {}_2F_1\left(\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}\middle|\frac{1+x}{2}\right). \tag{5}$$

$$T_n(x) = \tfrac{1}{2}n \sum_{k=0}^{[n/2]} \frac{(-)^k (n-k-1)!}{k!\,(n-2k)!}(2x)^{n-2k}, \qquad n = 1, 2, \ldots. \tag{6}$$

$$T_{2n}(x) = (-)^n\, {}_2F_1\left(\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}\middle| x^2\right) = {}_2F_1\left(\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}\middle| 1-x^2\right). \tag{7}$$

$$T_{2n}(x) = T_n(2x^2-1). \tag{8}$$

$$T_{2n+1}(x) = (-)^n(2n+1)\,x\, {}_2F_1\left(\begin{matrix}-n, n+1\\ \frac{3}{2}\end{matrix}\middle| x^2\right) = x\, {}_2F_1\left(\begin{matrix}-n, n+1\\ \frac{1}{2}\end{matrix}\middle| 1-x^2\right). \tag{9}$$

$$T_n(1) = 1, \qquad T_n(-1) = (-)^n, \qquad T_{2n}(0) = (-)^n, \qquad T_{2n+1}(0) = 0. \tag{10}$$

$$2^n(\tfrac{1}{2})_n T_n(x) = (-)^n(1-x^2)^{1/2}\frac{d^n}{dx^n}(1-x^2)^{n-1/2}. \tag{11}$$

$$(1-x^2)\frac{d^2T_n(x)}{dx^2} - \frac{x\,dT_n(x)}{dx} + n^2T_n(x) = 0. \tag{12}$$

$$(1-x^2)\frac{dT_n(x)}{dx} = n[T_{n-1}(x) - xT_n(x)]. \tag{13}$$

If

$$y_n(x) = AT_n(x) + BU_n(x),$$

where A and B are constants independent of n and x, then

$$\begin{aligned} &y_{n+1}(x) = 2xy_n(x) - y_{n-1}(x), && n > 0,\\ &T_1(x) = xT_0(x) = x, && U_1(x) = 2xU_0(x) = 2x, \end{aligned} \tag{14}$$

$$\begin{aligned} y_{2n+2}(x) &= 2(2x^2-1)\,y_{2n}(x) - y_{2n-2}(x) \qquad n>0, \\ T_2(x) &= (2x^2-1)\,T_0(x) = 2x^2-1, \\ U_2(x) &= (4x^2-1)\,U_0(x) = 4x^2-1 \end{aligned} \tag{15}$$

$$\begin{aligned} y_{2n+3}(x) &= 2(2x^2-1)\,y_{2n+1}(x) - y_{2n-1}(x), \qquad n>0, \\ T_3(x) &= 2(2x^2-1)\,T_1(x) - T_1(x) = (4x^2-3)\,T_1(x) = 4x^3-3x, \\ U_3(x) &= 2(2x^2-1)\,U_1(x) = 8x^3-4x \end{aligned} \tag{16}$$

$$2T_m(x)\,T_n(x) = T_{m+n}(x) + T_{|m-n|}(x) \tag{17}$$

$$2(x^2-1)\,U_{m-1}(x)\,U_{n-1}(x) = T_{m+n}(x) - T_{|m-n|}(x) \tag{18}$$

$$2T_m(x)\,U_{n-1}(x) = U_{n+m-1}(x) + U_{n-m-1}(x), \qquad n>m \tag{19}$$

$$2T_n(x)\,U_{m-1}(x) = U_{n+m-1}(x) - U_{n-m-1}(x), \qquad n>m. \tag{20}$$

$$T_m(x)\,U_n(x) - U_m(x)\,T_n(x) = T_1(x)\,U_{n-m-1}(x) \qquad n>m \tag{21}$$

$$x^m = 2^{1-m}\sum_{k=0}^{[m/2]}\binom{m}{k}T_{m-2k}(x), \qquad m>0 \tag{22}$$

$$x^m T_n(x) = 2^{-m}\sum_{k=0}^{m}\binom{m}{k}T_{|n+m-2k|}(x) \tag{23}$$

$$\frac{dT_{2n}(x)}{dx} = 4n\sum_{k=0}^{n-1}T_{2k+1}(x) \tag{24}$$

$$\frac{dT_{2n+1}(x)}{dx} = (2n+1) + 2(2n+1)\sum_{k=1}^{n}T_{2k}(x) \tag{25}$$

$$x\frac{dT_{2n}(x)}{dx} = 2n(T_0(x)+T_{2n}(x)) + 4n\sum_{k=1}^{n-1}T_{2k}(x) \tag{26}$$

$$x\frac{dT_{2n+1}(x)}{dx} = 2(2n+1)\sum_{k=0}^{n-1}T_{2k+1}(x) + (2n+1)\,T_{2n+1}(x) \tag{27}$$

$$x^2\frac{dT_{2n}(x)}{dx} = 4n\left[\sum_{k=0}^{n-2}T_{2k+1}(x) + \tfrac{3}{4}T_{2n-1}(x) + \tfrac{1}{4}T_{2n+1}(x)\right] \tag{28}$$

$$x^2\frac{dT_{2n+1}(x)}{dx} = 2(2n+1)\left[\tfrac{1}{2}T_0(x) + \sum_{k=1}^{n-1}T_{2k}(x) + \tfrac{3}{4}T_{2n}(x) + \tfrac{1}{4}T_{2n+2}(x)\right],$$

$$n \geqslant 1, \qquad x^2\frac{dT_1(x)}{dx} = \tfrac{1}{2}[T_0(x)+T_2(x)] \tag{29}$$

$$\frac{d^2T_{2n}(x)}{dx^2} = 4n^3T_0(x) + 8n\sum_{k=1}^{n-1}(n^2-k^2)\,T_{2k}(x). \tag{30}$$

$$\frac{d^2T_{2n+1}(x)}{dx^2} = 4(2n+1)\sum_{k=0}^{n-1}(n-k)(n+1+k)\,T_{2k+1}(x),$$

$$n \geqslant 1, \qquad \frac{d^2T_1(x)}{dx^2} = 0. \tag{31}$$

$$x\frac{d^2T_{2n}(x)}{dx^2} = 4n\sum_{k=0}^{n-1}\{2n^2-k^2-(k+1)^2\}\,T_{2k+1}(x). \tag{32}$$

$$x\frac{d^2T_{2n+1}(x)}{dx^2} = 2(2n+1)\left[n(n+1)\,T_0(x) + 2\sum_{k=1}^{n}(n^2+n-k^2)\,T_{2k}(x)\right]. \tag{33}$$

$$x^2\frac{d^2T_{2n}(x)}{dx^2} = 2n\left[(2n^2-1)T_0(x) + 2\sum_{k=1}^{n-1}(2n^2-2k^2-1)T_{2k}(x) + (2n-1)T_{2n}(x)\right]. \tag{34}$$

$$x^2\frac{d^2T_{2n+1}(x)}{dx^2} = 2(2n+1)\left[\sum_{k=0}^{n-1}(2n^2+2n-2k^2-2k-1)\,T_{2k+1}(x) + nT_{2n+1}(x)\right],$$

$$n \geqslant 1. \tag{35}$$

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^{-1/2}T_n(x)\,T_m(x)\,dx &= 0 && \text{if} \quad m \neq n,\\ &= \pi/2 && \text{if} \quad m = n \neq 0,\\ &= \pi && \text{if} \quad m = n = 0.\end{aligned} \tag{36}$$

$$\int_{-1}^{1}(1-x^2)^{1/2}U_n(x)\,U_m(x)\,dx = (\pi/2)\delta_{mn}. \tag{37}$$

$$\int_{-1}^{1}(x-y)^{-1}(1-x^2)^{-1/2}[T_n(x)-T_n(y)]\,dx = \pi U_{n-1}(y). \tag{38}$$

$$\text{P.V.}\int_{-1}^{1}(x-y)^{-1}(1-x^2)^{-1/2}T_n(x)\,dx = \pi U_{n-1}(y), \qquad -1 < y < 1. \tag{39}$$

$$\text{P.V.}\int_{-1}^{1}(x-y)^{-1}(1-x^2)^{1/2}U_{n-1}(x)\,dx = -\pi T_n(y), \qquad -1 < y < 1. \tag{40}$$

$$\int_{-1}^{1}(z-x)^{-1}(1-x^2)^{-1/2}T_n(x)\,dx = \frac{\pi e^{-n\varphi}}{\sinh\varphi}, \tag{41}$$

$$\int_{-1}^{1}(z-x)^{-1}(1-x^2)^{1/2}U_{n-1}(x)\,dx = \pi e^{-n\varphi}, \qquad n > 0,$$

$$z = \cosh\varphi, \qquad e^{\varphi} = z \pm (z^2-1)^{1/2} \tag{42}$$

and the sign is chosen so that $|e^{\sigma}| > 1$

$$\int_0^x T_{2n}(t)\,dt = \frac{1}{2}\left[\frac{T_{2n+1}(x)}{2n+1} - \frac{T_{2n-1}(x)}{2n-1}\right] \tag{43}$$

$$\int_0^x T_{2n+1}(t)\,dt = \frac{1}{2}\left[\frac{T_{2n+2}(x)}{2n+2} - \frac{T_{2n}(x)}{2n}\right] + \frac{(-)^n}{4}\left(\frac{1}{n} + \frac{1}{n+1}\right), \qquad n > 0,$$

$$\int_0^x T_1(t)\,dt = \tfrac{1}{4}[T_2(x) + T_0(x)] \tag{44}$$

$$(n+1)\int_0^x U_n(t)\,dt = T_{n+1}(x) - T_{n+1}(0) \tag{45}$$

$$\int_0^x\int_0^t T_{2n}(u)\,du\,dt = \frac{T_{2n+2}(x)}{4(2n+1)(2n+2)} - \frac{T_{2n}(x)}{2(4n^2-1)} + \frac{T_{2n-2}(x)}{4(2n-1)(2n-2)}$$
$$+ \frac{(-)^n}{4(n^2-1)}, \qquad n > 1,$$

$$\int_0^x\int_0^t T_0(u)\,du\,dt = \tfrac{1}{4}[T_0(x) + T_2(x)],$$

$$\int_0^x\int_0^t T_2(u)\,du\,dt = -\frac{3T_0(x)}{16} - \frac{T_2(x)}{6} + \frac{T_4(x)}{48} \tag{46}$$

$$\int_0^x\int_0^t T_{2n+1}(u)\,du\,dt = \frac{T_{2n+3}(x)}{4(2n+2)(2n+3)} - \frac{T_{2n+1}(x)}{2(2n+2)(2n)} + \frac{T_{2n-1}(x)}{4(2n)(2n-1)}$$
$$+ \frac{(-)^n(2n+1)}{4n(n+1)}\,T_1(x) \qquad n > 0,$$

$$\int_0^x\int_0^t T_1(u)\,du\,dt = \tfrac{1}{24}[3T_1(x) + T_3(x)] \tag{47}$$

8.5.2 The Polynomials $T_n^*(x)$ and $U_n^*(x)$

The shifted Chebyshev polynomials of the first and second kinds are denoted by $T_n^*(x)$ and $U_n^*(x)$, respectively. We have the connecting relations

$$T_n^*(x) = T_n(2x-1), \qquad T_n(x) = T_n^*[(1+x)/2] \tag{1}$$

$$T_{2n}(x) = T_n^*(x^2), \tag{2}$$

$$U_n^*(x) = U_n(2x-1), \qquad U_n(x) = U_n^*[(1+x)/2] \tag{3}$$

$$U_{2n+1}(x) = 2xU_n^*(x^2) \tag{4}$$

Identities for the shifted polynomials are readily derived from those of the unshifted polynomials. We omit further properties of $U_n^*(x)$. For

the applications, it is convenient to enumerate some relations involving $T_n^*(x)$.

$$T_n^*(x) = {}_2F_1\left(\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}\middle| 1-x\right) = (-)^n\, {}_2F_1\left(\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}\middle| x\right). \tag{5}$$

$$T_n^*(1-x) = (-)^n T_n^*(x). \tag{6}$$

$$T_n^*(1) = 1, \qquad T_n^*(0) = (-)^n. \tag{7}$$

$$T_{n+1}^*(x) = 2(2x-1)\, T_n^*(x) - T_{|n-1|}^*(x). \tag{8}$$

$$\begin{aligned}\int_0^1 \frac{T_n^*(x)\, T_m^*(x)}{[x(1-x)]^{1/2}}\, dx &= 0 \qquad \text{if} \quad m \neq n,\\ &= \tfrac{1}{2}\pi \qquad \text{if} \quad m = n \neq 0,\\ &= \pi \qquad \text{if} \quad m = n = 0.\end{aligned} \tag{9}$$

$$x^m = 2^{-2m}\left[\binom{2m}{m} T_0^*(x) + 2\sum_{k=1}^{m}\binom{2m}{m-k} T_k^*(x)\right]. \tag{10}$$

$$x^m T_n^*(x) = 2^{-2m}\sum_{k=0}^{2m}\binom{2m}{k} T_{|n+m-k|}^*(x). \tag{11}$$

$$\frac{dT_{2n}^*(x)}{dx} = 8n\sum_{k=0}^{n-1} T_{2k+1}^*(x). \tag{12}$$

$$\frac{dT_{2n+1}^*(x)}{dx} = 2(2n+1)\, T_0^*(x) + 4(2n+1)\sum_{k=1}^{n} T_{2k}^*(x). \tag{13}$$

$$x\frac{dT_n^*(x)}{dx} = 2n\left[\tfrac{1}{2}\{T_0^*(x) + T_n^*(x)\} + \sum_{k=1}^{n-1} T_k^*(x)\right]. \tag{14}$$

$$\begin{aligned}x^2\frac{dT_n^*(x)}{dx} = 2n\Bigg[&\tfrac{1}{2}T_0^*(x) + \sum_{k=1}^{n-2} T_k^*(x) + \tfrac{7}{8}T_{n-1}^*(x) + \tfrac{1}{2}T_n^*(x)\\ &+ \tfrac{1}{8}T_{n+1}^*(x)\Bigg], \qquad n \geqslant 2;\end{aligned}$$

$$x^2\frac{dT_1^*(x)}{dx} = \tfrac{3}{4}T_0^*(x) + T_1^*(x) + \tfrac{1}{4}T_2^*(x), \qquad x^2\frac{dT_0^*(x)}{dx} = 0. \tag{15}$$

$$\frac{d^2T_{2n}^*(x)}{dx^2} = 16n^3T_0^*(x) + 32n\sum_{k=1}^{n-1}(n^2-k^2)\, T_{2k}^*(x). \tag{16}$$

$$\frac{d^2T_{2n+1}^*(x)}{dx^2} = 16(2n+1)\sum_{k=0}^{n-1}(n-k)(n+1+k)\, T_{2k+1}^*(x). \tag{17}$$

$$x\frac{d^2T^*_{2n}(x)}{dx^2} = 8n\left[n^2T^*_0(x) + 2\sum_{k=1}^{n-1}(n^2-k^2)\,T^*_{2k}(x)\right.$$

$$\left.+\sum_{k=0}^{n-1}\{2n^2-k^2-(k+1)^2\}\,T^*_{2k+1}(x)\right] \tag{18}$$

$$x\frac{d^2T^*_{2n+1}(x)}{dx^2} = 8(2n+1)\left[\frac{n(n+1)}{2}\,T^*_0(x) + \sum_{k=1}^{n}(n^2+n-k^2)\,T^*_{2k}(x)\right.$$

$$\left.+\sum_{k=0}^{n-1}(n-k)(n+1+k)\,T^*_{2k+1}(x)\right] \tag{19}$$

$$x^2\frac{d^2T^*_{2n}}{dx^2} = 2n\left[(4n^2-1)\,T^*_0(x) + 2\sum_{k=1}^{n-1}\{4(n^2-k^2)-1\}\,T^*_{2k}(x)\right.$$

$$\left.+(2n-1)\,T^*_{2n}(x) + 4\sum_{k=0}^{n-1}\{2n^2-k^2-(k+1)^2\}\,T^*_{2k+1}(x)\right] \tag{20}$$

$$x^2\frac{d^2T^*_{2n+1}(x)}{dx^2} = (2n+1)\left[4n(n+1)\,T^*_0(x) + 8\sum_{k=1}^{n}(n^2+n-k^2)\,T^*_{2k}(x)\right.$$

$$\left.+2\sum_{k=0}^{n-1}(4n^2+4n-4k^2-4k-1)\,T^*_{2k+1}(x) + 2nT^*_{2n+1}(x)\right] \tag{21}$$

$$\int_0^x T^*_n(t)\,dt = \frac{1}{4}\left[\frac{T^*_{n+1}(x)}{n+1} - \frac{T^*_{|n-1|}(x)}{n-1}\right] - \frac{(-)^n}{2(n^2-1)} \qquad n = 0, 2, 3$$

$$\int_0^x T^*_1(t)\,dt = \tfrac{1}{8}[T^*_2(x) - T^*_0(x)] \qquad \int_0^x T^*_0(t)\,dt = \tfrac{1}{2}[T^*_1(x) + T^*_0(x)] \tag{22}$$

$$\int_0^x\int_0^t T^*_n(u)\,du\,dt = \frac{T^*_{n+2}(x)}{16(n+2)(n+1)} - \frac{T^*_n(x)}{8(n^2-1)} + \frac{T^*_{n-2}(x)}{16(n-1)(n-2)}$$

$$-\frac{(-)^nT^*_1(x)}{4(n^2-1)} - \frac{(-)^nT^*_0(x)}{4(n^2-4)} \qquad n > 2$$

$$\int_0^x\int_0^t T^*_0(u)\,du\,dt = \tfrac{1}{2}x^2 = \tfrac{1}{10}[3T^*_0(x) + 4T^*_1(x) + T^*_2(x)] \tag{23}$$

$$\int_0^x\int_0^t T^*_1(u)\,du\,dt = \tfrac{1}{96}[-8T^*_0(x) - 9T^*_1(x) + T^*_3(x)]$$

$$\int_0^x\int_0^t T^*_2(u)\,du\,dt = \tfrac{1}{192}[-9T^*_0(x) - 16T^*_1(x) - 8T^*_2(x) + T^*_4(x)]$$

8.5.3. Minimax Properties and Comparison with the Truncated Mean Square Approximation

Let $f(x)$ and $s(x)$ be real and continuous in $[a, b]$ and consider

$$Q_{n,m}(x) = \frac{s(x)\, q_n(x)}{p_m(x)}, \qquad q_n(x) = \sum_{k=0}^{n} a_k x^k, \qquad p_m(x) = \sum_{k=0}^{m} b_k x^k \tag{1}$$

where m and n are given. The problem proposed by Chebyshev is that of finding real numbers a_k and b_k so that the maximum deviation of $Q_{n,m}(x)$ from $f(x)$ is least. Such an approximation is called best in the Chebyshev sense. We discuss very briefly only the special case $s(x) = 1$ and $m = 0$. For the general case, related topics, and further considerations of the special case just noted, see Akhiezer (1956), Walsh (1960), Todd (1962), P. J. Davis (1963), J. R. Rice (1964), Meinardus (1964), Cheney (1966), and Handscomb (1966).

A striking feature of the Chebyshev polynomial $T_n(x)$ is manifested by the following.

Theorem 1. *Of all polynomials of degree n with leading coefficient unity, that which deviates least from zero in $[-1, 1]$ is the Chebyshev polynomial $2^{1-n}T_n(x)$.*

PROOF. Let $q_n(x)$ be given as in (1) with $a_0 = 1$. Let

$$\nu(q_n) = \max_{-1 \leqslant x \leqslant 1} |\, q_n(x)|.$$

Then we want to prove that $\nu(q_n) \geqslant 2^{1-n}$ and equality obtains if and only if $q_n(x) = T_n(x)$. Suppose the theorem is false. Then there is a polynomial with leading coefficient unity, call it $q_n^*(x)$, such that $\nu(q_n^*) < 2^{1-n}$. Put

$$r(x) = q_n^*(x) - 2^{1-n}T_n(x),$$

which is a polynomial of degree $(n - 1)$. It cannot vanish identically, for if it did, we would have $q_n^*(x) = 2^{1-n}T_n(x)$, whence $\nu(q_n^*) = 2^{1-n}$, which contradicts our assumption. It is clear from 8.5.1(2) that

$$T_n(x_\beta) = (-)^\beta, \qquad x_\beta = \cos \pi\beta/n, \qquad \beta = 0, 1, \ldots, n,$$
$$|\, T_n(x)| \leqslant 1, \qquad -1 \leqslant x \leqslant 1.$$

Now consider the values of $r(x)$ at the points x_β. Since $|\, q_n^*(x)| < 2^{1-n}$, $\operatorname{sign}[r(x_\beta)] = (-)^\beta$. It follows that $r(x)$ must vanish between x_β and $x_{\beta+1}$. That is, $r(x)$ must vanish n times in $[-1, 1]$. But this is impossible since $r(x)$ is a polynomial of degree $(n - 1)$. Thus $r(x)$ must vanish identically and the theorem follows.

Some further theorems concerning polynomials of best approximation in the Chebyshev sense to a continuous function follow For proof, see the references already noted

Theorem 2. *If $f(x)$ is continuous in $[a, b]$, then there exists a polynomial of degree n, $Q_n(x)$, such that*

$$E_n(f) = \max_{a \leqslant x \leqslant b} |f(x) - Q_n(x)| < \max_{a \leqslant x \leqslant b} |f(x) - r_n(x)|$$

for any $r_n(x) \neq Q_n(x)$ $Q_n(x)$ is called the best approximation of degree n

Theorem 3. *There are $(n + 2)$ points $a \leqslant x_0 < x_1 < \quad < x_n < x_{n+1} \leqslant b$ such that*

$$|f(x_i) - Q_n(x_i)| = E_n(f), \qquad i = 0, 1, \quad , n + 1,$$

and $f(x_i) - Q_n(x_i)$ alternates in sign for $i = 0, 1, \quad , n + 1$

Theorem 4. *If there exist $(n + 2)$ points*

$$a \leqslant x_0 < x_1 < \quad < x_n < x_{n+1} \leqslant b$$

such that

$$f(x_i) - r_n(x_i) = \omega_i, \qquad i = 0 \ 1, \quad , n + 1,$$

where the ω_i alternate in sign, then

$$E_n(f) \geqslant \min(|\omega_0|, |\omega_1|, \quad , |\omega_{n+1}|)$$

Further, if

$$\omega_i = \max_{a \leqslant x \leqslant b} |f(x_i) - r_n(x_i)| \qquad i = 0, 1, \quad , n + 1,$$

then $r_n(x) = Q_n(x)$

Theorem 5. *If $Q_n(x)$ is the best polynomial approximation to $f(x)$ of degree n, then*

$$f(x) - Q_n(x) = \omega T_{n+1}(x), \qquad \omega = \max_{a \leqslant x \leqslant b} |f(x) - Q_n(x)|$$

Only in a few cases it is possible to obtain $Q_n(x)$ in closed form In this connection, see Rivlin (1962), whose work includes that of Bernstein (1926), Hornecker (1958), and Talbot (1962) as special cases See also Akhiezer (1956, p 64) for an example of (1) with $m = n$

Algorithms for the computation of $Q_{n,m}(x)$ have been studied by many authors. See, for example, the references cited following (1). The algorithms are, in general, not finite and numerical values of the pertinent parameters are produced by iterative processes.

Next we turn to some results which enable us to compare the best approximation in the Chebyshev sense with the truncated mean square approximation, each with the same number of terms and over the same interval $[-1, 1]$. Let

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x), \qquad f_n(x) = \sum_{k=0}^{n} a_k T_k(x),$$

$$s_n(x) = f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k T_k(x), \tag{2}$$

$$S_n = \max_{-1 \leqslant x \leqslant 1} |s_n(x)| \leqslant \sum_{k=n+1}^{\infty} |a_k| = S_n^*.$$

Let $Q_n(x)$ be the best polynomial approximation of degree n to $f(x)$ in $-1 \leqslant x \leqslant 1$. Let

$$E_n = \max_{-1 \leqslant x \leqslant 1} |f(x) - Q_n(x)|. \tag{3}$$

Theorem 6.

$$A_n \leqslant E_n \leqslant S_n \leqslant \sum_{k=n+1}^{\infty} |a_k| = S_n^*, \tag{4}$$

where A_n is any one of the values

$$A_{n,1} = 2^{-1/2}\left(\sum_{k=n+1}^{\infty} a_k^2\right)^{1/2}, \qquad A_{n,2} = 2^{-1/2}|a_{n+1}|, \qquad A_{n,3} = \left|\sum_{k=1}^{\infty} a_{(2k-1)(n+1)}\right|, \tag{5}$$

and if the a_k's are nonnegative, we can replace $A_{n,3}$ by a_{n+1}.

PROOF. Note that $E_n \leqslant S_n$ follows from Theorem 2. The inequality involving $A_{n,2}$ follows from that involving $A_{n,1}$, so it is sufficient to consider the latter. From Theorem 1 of 8.1,

$$\int_{-1}^{1} \frac{s_n^2(x)}{(1-x^2)^{1/2}}\,dx \leqslant \int_{-1}^{1} \frac{\{f(x) - Q_n(x)\}^2}{(1-x^2)^{1/2}}\,dx.$$

Now the integral on the right cannot exceed πE_n^2 and the integral on the left is easily evaluated using the orthogonality properties of $T_n(x)$.

Thus the stated inequality readily emerges The weaker inequality with $2^{1/2}$ in $A_{n,2}$ replaced by $\pi/4$ has been proved by Blum and Curtis (1961)

The inequality involving $A_{n,3}$ is a classic result due to de la Vallee Poussin (1952) Let us write

$$F = \tfrac{1}{2}[\{f(\cos\varphi_0) - Q_n(\cos\varphi_0)\}\cos(n+1)\varphi_0 + \{f(\cos\varphi_{n+1}) - Q_n(\cos\varphi_{n+1})\}\cos(n+1)\varphi_{n+1}] + \sum_{j=1}^{n}\{f(\cos\varphi_j) - Q_n(\cos\varphi_j)\}\cos(n+1)\varphi_j,$$

where $\varphi_\alpha = \alpha\pi/(n+1)$, $\alpha = 0, 1, \ldots, n+1$ Clearly $|F| \leqslant (n+1)E_n$ Now let

$$Q_n(\cos\varphi_j) = \sum_{k=0}^{n} b_k \cos k\varphi_j$$

From 8 5 4(7), $F = 0$ when $f(x) \equiv 0$ Thus, $F = \sum_{k=0}^{\infty} a_k V_{k,n+1}$, where $V_{k,n+1}$ is given by 8 5 4(7) (There replace j, k and n by k, $n+1$ and $n+1$, respectively) In view of 8 5 4(8), $V_{k,n+1}$ is zero unless k is an odd multiple of $n+1$, in which event it has the value $n+1$ So $F = (n+1)\sum_{k=1}^{\infty} a_{(2k-1)(n+1)}$ and the result readily follows This completes the proof of Theorem 6

Given the coefficients in the Chebyshev expansion for $f(x)$ as in (2), the best approximation in the Chebyshev sense can often be determined by use of formulas due to Hornecker (1958) However, the gain in accuracy is usually slight For a numerical illustration, see Clenshaw (1962) In particular, for the expansion of ${}_pF_q(a_p, b_q, z\omega)$ [see 8 4 3(18)] the results 7 3(7–9) show that for n sufficiently large, $A_{n,3}$ is very near to $|a_{n+1}|$ and $|a_{n+1}|$ is a good approximation to S_n^* In the applications as evidenced by the numerous expansions in series of Chebyshev polynomials which we give in Chapter XVII, we see that $A_{n,3}$ is indeed extremely close to $|a_{n+1}|$ and S_n^* seldom exceeds $2|a_{n+1}|$ for n sufficiently large

It is pertinent to discuss some further aspects of the relative merits of the best approximation in the Chebyshev sense, $Q_n(x)$, and the best in the mean square sense, $f_n(x)$ As previously remarked, $Q_n(x)$ is known in closed form for very few functions In general, finite algorithms are not known and one must use iterative processes Here a rather extensive table of $f(x)$ must be available In general, there is no known connection between $Q_n(x)$ for the interval $[a, b]$ and $Q_m(x)$ for the interval $[c, d]$ For many special functions of mathematical physics, the coefficients which define $f_n(x)$ are readily expressed in closed form and are easily

determined by use of recursion formulas, see Chapters IX and XII. For the most part, we need not postulate the existence of a mathematical table for $f(x)$. Further, the approximations $f_n(x)$ are much like that of a Taylor series in that if we are given $f_n(x)$ and desire greater accuracy, we need only add more terms.

8.5.4. Orthogonality Properties of Chebyshev Polynomials with Respect to Summation

We first prove the following

Lemma 1. *Let*

$$T_n(x_\alpha) = \cos n\theta_\alpha = 0,$$
$$x_\alpha = \cos\theta_\alpha, \qquad \theta_\alpha = (\pi/2n)(2\alpha+1), \qquad \alpha = 0, 1, \ldots, n-1,$$
$$U_{j,k} = \sum_{\alpha=0}^{n-1} T_j(x_\alpha)\, T_k(x_\alpha) = \sum_{\alpha=0}^{n-1} \cos j\theta_\alpha \cos k\theta_\alpha .$$

Then

$$\begin{aligned} U_{j,k} &= 0 && \text{if } j, k < n, \quad j \neq k, \\ &= n/2 && \text{if } j = k, \quad 0 < j < n, \\ &= n && \text{if } j = k = 0. \end{aligned} \tag{1}$$

We also have some further properties of $U_{j,k}$. For convenience put $m = j + k$, $p = |j - k|$, and let s and t be positive integers. Then

$$\begin{aligned} U_{j,0} &= (-)^s n && \text{if } j = 2sn, \\ &= 0 && \text{if } j \neq 2sn. \\ U_{j,j} &= n && \text{if } j = 2sn, \\ &= n/2 && \text{if } j \neq sn, \\ &= 0 && \text{if } j = (2s+1)n. \\ U_{j,k} &= 0 && \text{if } j \neq k,\ m \neq 2sn,\ p \neq 2tn, \\ &= n && \text{if } j \neq k,\ m = 2sn,\ p = 2tn, \text{ and } s \text{ and } t \text{ are both even}, \\ &= -n && \text{if } j \neq k,\ m = 2sn,\ p = 2tn, \text{ and } s \text{ and } t \text{ are both odd}, \\ &= 0 && \text{if } j \neq k,\ m = 2sn,\ p = 2tn, \text{ and } s \text{ and } t \text{ are of opposite parity}, \\ &= (-)^s n/2 && \text{if } j \neq k,\ m = 2sn,\ p \neq 2tn, \\ &= (-)^t n/2 && \text{if } j \neq k,\ m \neq 2sn,\ p = 2tn. \end{aligned} \tag{2}$$

PROOF We prove only (1) The result is trivial if $j = k = 0$ Now

$$U_{jk} = \frac{1}{2}\left(\sum_{\alpha=0}^{n-1} \cos(j+k)\theta_\alpha + \sum_{\alpha=0}^{n-1} \cos(j-k)\theta_\alpha\right)$$

$$= \tfrac{1}{2}R\left[\exp\left(\frac{i\pi m}{2n}\right)\sum_{\alpha=0}^{n-1}\exp\left(\frac{i\alpha\pi m}{n}\right)\right] + \tfrac{1}{2}R\left[\exp\left(\frac{i\pi p}{2n}\right)\sum_{\alpha=0}^{n-1}\exp\left(\frac{i\alpha\pi p}{n}\right)\right]$$

$$= \tfrac{1}{2}R(A_1 + A_2),$$

$$A_j = \frac{e^{i\varphi}[e^{2i\varphi_j n} - 1]}{e^{2i\varphi_j} - 1} \qquad \varphi_1 = \frac{m\pi}{2n}, \qquad \varphi_2 = \frac{p\pi}{2n}$$

Note that if $j = k$, $0 < j < n$, $p = 0$, and $A_2 = n$ Thus for this assignment of j and k or if $j, k < n$, $j \neq k$, it is sufficient to prove that $R(A_1) = 0$ Now (dropping subscripts)

$$A = \frac{e^{i\varphi}(e^{2i\varphi n} - 1)(\cos 2\varphi - 1 - i\sin 2\varphi)}{2(1 - \cos 2\varphi)}$$

Clearly $m/2n < 1$ and so $1 - \cos 2\varphi \neq 0$ A straightforward computation shows that

$$R(A) = \frac{[(-)^m - 1]}{2(1 - \cos 2\varphi)}(\cos\varphi\cos 2\varphi + \sin\varphi\sin 2\varphi - \cos\varphi) = 0,$$

which completes the proof of (1)

Next we establish the following expansion formula

Theorem 1. *If $f_n(x)$ is an approximation to $f(x)$,*

$$f_n(\cos\theta) = \tfrac{1}{2}d_0 + \sum_{k=1}^{n-1} d_k \cos k\theta \qquad \text{or} \qquad f_n(x) = \tfrac{1}{2}d_0 + \sum_{k=1}^{n-1} d_k T_k(x)$$

then

$$d_k = (2/n)\sum_{\alpha=0}^{n-1} f(\cos\theta_\alpha)\cos k\theta_\alpha = (2/n)\sum_{\alpha=0}^{n-1} f(x_\alpha)\,T_k(x_\alpha) \tag{3}$$

PROOF In $f_n(\cos\theta)$ replace θ by θ_α, multiply this equation by $\cos j\theta_\alpha$ and sum on α from 0 to $(n-1)$ The desired result follows from (1)

We can use our developments to get a formula for the error in this approximation process. We suppose that $f'(x)$ is continuous in $[-1, 1]$ except for a finite number of bounded jumps. Then from Theorems 1 and 2 of 8.3, $f(x)$ can be expanded in a convergent series as

$$f(x) = \tfrac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k T_k(x),$$

$$c_k = (2/\pi)\int_{-1}^{1} \frac{f(x)\,T_k(x)}{(1-x^2)^{1/2}}\,dx = (2/\pi)\int_0^{\pi} f(\cos\theta)\cos k\theta\,d\theta. \tag{4}$$

We first obtain a connection between the d_k's and the c_k's. Thus combine d_k as in (3) with $f(x)$ above and use (1), (2) to get

Lemma 2.

$$d_0 = c_0 + 2\sum_{r=1}^{\infty} (-)^r c_{2rn}$$

$$d_k = c_k + \sum_{r=1}^{\infty} (-)^r c_{2rn-k} + \sum_{r=1}^{\infty} (-)^r c_{2rn+k}, \qquad k = 1, 2, \ldots, n-1. \tag{5}$$

Theorem 2. *If*

$$\epsilon_n(x) = f(x) - f_n(x),$$

where $f(x)$ and $f_n(x)$ are given by (4) *and* (3), *respectively, then*

$$\begin{aligned}\epsilon_n(\cos\theta) = {} & \cos n\theta\left\{c_n + 2\sum_{r=1}^{2n-1} c_{n+r}\cos r\theta + c_{3n}\cos 2n\theta\right\} \\ & - \sin 2n\theta\left\{c_{3n}\sin n\theta + 2\sum_{r=1}^{2n-1} c_{3n+r}\sin(n+r)\theta + c_{5n}\sin 3n\theta\right\} \\ & + \cos 3n\theta\left\{c_{5n}\cos 2n\theta + 2\sum_{r=1}^{2n-1} c_{5n+r}\cos(2n+r)\theta + c_{7n}\cos 4n\theta\right\} - \cdots,\end{aligned} \tag{6}$$

$$\epsilon_n(\cos\theta_\alpha) = 0,$$

$$\epsilon_n(\cos\theta) \sim c_n \cos n\theta[1 + (2c_{n+1}/c_n)\cos\theta].$$

REMARK. See the comments after 8.4.3(12).

PROOF Form $\epsilon_n(x)$ and use (5) Then

$$\epsilon_n(\cos\theta) = \sum_{k=n}^{\infty} c_k T_k(x) - \tfrac{1}{2}\sum_{r=1}^{\infty}(-)^r c_{2nr}$$

$$+ \sum_{r=1}^{n-1} c_{n+r}\{\cos(n+r)\theta + \cos(n-r)\theta\}$$

$$+ \sum_{r=1}^{n-1} c_{2n+r}\{\cos(2n+r)\theta + \cos r\theta\}$$

$$+ \sum_{r=1}^{n-1} c_{3n+r}\{\cos(3n+r)\theta - \cos(n-r)\theta\}$$

$$+ \sum_{r=1}^{n-1} c_{4n+r}\{\cos(4n+r)\theta - \cos r\theta\}$$

$$+ \sum_{r=1}^{n-1} c_{5n+r}\{\cos(5n+r)\theta + \cos(n-r)\theta\}$$

$$+ \sum_{r=1}^{n-1} c_{6n+r}\{\cos(6n+r)\theta + \cos r\theta\} + \quad ,$$

and the desired result follows upon using an elementary identity

Obviously in (3) we restrict x so that $-1 \leqslant x \leqslant 1$ Observe that each x_α lies in the interior of this interval In numerous applications, it is useful to have a curve fit which uses the end point $x = \pm 1$ and the points which lie midway between the points θ_α Here again the Chebyshev polynomials possess an orthogonality property which leads to an expansion formula based on the abscissas just named We state lemmas and theorems analogous to those above Proofs are omitted

Lemma 3. *Let*

$$x_\alpha = \cos\varphi_\alpha \qquad \varphi_\alpha = \alpha\pi/n, \qquad \alpha = 0\ 1, \ \ , n,$$

$$V_{j\,k} = \tfrac{1}{2}\{T_j(x_0)\,T_k(x_0) + T_j(x_n)\,T_k(x_n)\} + \sum_{\alpha=1}^{n-1} T_j(x_\alpha)\,T_k(x_\alpha)$$

$$= \tfrac{1}{2}\{\cos j\varphi_0 \cos k\varphi_0 + \cos j\varphi_n \cos k\varphi_n\} + \sum_{j=1}^{n-1} \cos j\varphi_\alpha \cos k\varphi_\alpha$$

Then

$$\begin{aligned} V_{j,k} &= 0 && \text{if } j, k < n, \quad j \neq k, \\ &= n/2 && \text{if } \quad j = k, \quad 0 < j < n, \\ &= n && \text{if } \quad j = k = 0 \quad \text{or } j = k = n. \end{aligned} \tag{7}$$

Some further values of $V_{j,k}$ follow. It is convenient to put $m = j + k$, $p = |j - k|$, and let s and t be positive integers. Then

$$\begin{aligned} V_{j,0} &= n && \text{if } j = 2sn, \\ &= 0 && \text{if } j \neq 2sn. \\ V_{j,j} &= n && \text{if } j \neq 0, \quad j \neq n, \quad j = sn, \\ &= n/2 && \text{if } j \neq 0, \quad j \neq n, \quad j \neq sn. \\ V_{j,k} &= 0 && \text{if } j \neq k, \quad m \neq 2sn, \quad p \neq 2tn, \\ &= n && \text{if } j \neq k, \quad m = 2sn, \quad p = 2tn, \\ &= n/2 && \text{if } j \neq k, \quad m = 2sn, \quad p \neq 2tn, \\ &= n/2 && \text{if } j \neq k, \quad m \neq 2sn, \quad p = 2tn. \end{aligned} \tag{8}$$

Theorem 3. *If $f_n(x)$ is an approximation to $f(x)$,*

$$f_n(\cos\varphi) = \tfrac{1}{2}e_0 + \sum_{k=1}^{n-1} e_k \cos k\varphi + \tfrac{1}{2}e_n \cos n\varphi$$

or

$$f_n(x) = \tfrac{1}{2}e_0 + \sum_{k=1}^{n-1} e_k T_k(x) + \tfrac{1}{2}e_n T_n(x),$$

then

$$e_k = \frac{2}{n}\left[\frac{f(1) + (-)^k f(-1)}{2} + \sum_{\alpha=1}^{n-1} f(\cos\varphi_\alpha)\cos k\varphi_\alpha\right],$$

or (9)

$$e_k = \frac{2}{n}\left[\frac{f(1) + (-)^k f(-1)}{2} + \sum_{\alpha=1}^{n-1} f(x_\alpha)\, T_k(x_\alpha)\right].$$

Lemma 4.

$$e_0 = c_0 + 2\sum_{r=1}^{\infty} c_{2rn}, \qquad e_n = 2c_n + 2\sum_{r=1}^{\infty} c_{(2r+1)n},$$

$$e_k = c_k + \sum_{r=1}^{\infty} c_{2rn-k} + \sum_{r=1}^{\infty} c_{2rn+k}, \qquad k = 1, 2, \ldots, n-1. \tag{10}$$

Theorem 4 *If*

$$\delta_n(x) = f(x) - f_n(x),$$

where $f(x)$ and $f_n(x)$ are given by (4) *and* (9), *respectively, then*

$$\begin{aligned} \delta_n(\cos\varphi) &= -2\sin n\varphi \sum_{r=1}^{\infty} c_{n+r} \sin r\varphi, \\ \delta_n(\cos\varphi_\alpha) &= 0, \\ \delta_n(\cos\varphi) &\sim -2\sin n\varphi \sin\varphi\, c_{n+1}\left[1 + \frac{2c_{n+2}}{c_{n+1}}\cos\varphi\right] \end{aligned} \tag{11}$$

If c_k in (4) is evaluated approximately by the trapezoidal rule, then e_k is an approximation to c_k and (10) may be interpreted as the error in this process Similarly, if c_k is evaluated by what we call the modified trapezoidal rule, that is, if c_k is approximated by d_k [see (3)], then (5) may be interpreted as the error in this numerical integration scheme Note that $\frac{1}{2}(d_k + e_k)$ is an improved approximation for c_k, $k = 0, 1, \ldots, n$

In the above, we dealt with expansions in series of $T_n(x)$ We now give without proof analogous statements for expansions in series of Chebyshev polynomials of the second kind $U_n(x)$

Lemma 5. *Let*

$$U_n(x_\alpha) = \frac{\sin(n+1)\theta_\alpha}{\sin\theta_\alpha},$$

$$x_\alpha = \cos\theta_\alpha, \qquad \theta_\alpha = \frac{\pi}{2n}(2\alpha + 1) \qquad \alpha = 0, 1, \ldots, n-1,$$

$$W_{j,k} = \sum_{\alpha=0}^{n-1} (1 - x_\alpha^2)\, U_{j-1}(x_\alpha)\, U_{k-1}(x_\alpha) = \sum_{\alpha=0}^{n-1} \sin j\theta_\alpha \sin k\theta_\alpha$$

Then

$$\begin{aligned} W_{j,k} &= 0 && \text{if } j, k < n, \quad j \neq k, \\ &= n/2 && \text{if } j = k \quad 0 < j < n, \\ &= 0 && \text{if } j = 0 \quad \textit{for all } k \textit{ or if } k = 0 \textit{ for all } j \end{aligned} \tag{12}$$

Some further properties of $W_{j,k}$ and also of $X_{j,k}$ [see (18)] like (2) and (8) can be deduced, but we omit details

Theorem 5. *If $f_n(x)$ is an approximation to $f(x)$,*

$$f_n(\cos\theta) = \sum_{k=1}^{n-1} q_k \sin k\theta \qquad \text{or} \qquad f_n(x) = (1 - x^2)^{1/2} \sum_{k=1}^{n-1} q_k U_{k-1}(x)$$

then

$$q_k = \frac{2}{n}\sum_{\alpha=0}^{n-1} f(\cos\theta_\alpha)\sin k\theta_\alpha = \frac{2}{n}\sum_{\alpha=0}^{n-1}(1-x_\alpha^2)^{1/2} f(x_\alpha)\, U_{k-1}(x_\alpha). \tag{13}$$

Again we suppose that $f'(x)$ is defined as in the discussion surrounding (4). *Then*

$$f(x) = (1-x^2)^{1/2}\sum_{k=1}^{\infty} p_k U_{k-1}(x) = \sum_{k=1}^{\infty} p_k \sin k\theta,$$

$$p_k = \frac{2}{\pi}\int_{-1}^{1} f(x)\, U_{k-1}(x)\, dx = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\sin k\theta\, d\theta. \tag{14}$$

Lemma 6.

$$q_j = p_j - \sum_{r=1}^{\infty}(-)^r p_{2nr-j} + \sum_{r=1}^{\infty}(-)^r p_{2nr+j}, \qquad j = 1, 2, \ldots, n-1. \tag{15}$$

Theorem 6. *If*

$$\epsilon_n(x) = f(x) - f_n(x)$$

where $f(x)$ and $f_n(x)$ are given by (14) *and* (13), *respectively, then*

$$\begin{aligned} \epsilon_n(\cos\theta) &= p_n \sin n\theta + 2\cos n\theta \sum_{r=1}^{\infty} p_{n+r}\sin r\theta, \\ \epsilon_n(\cos\theta_\alpha) &= (-)^\alpha p_n, \\ \epsilon_n(\cos\theta) &\sim p_n \sin n\theta\left(1 + \frac{2p_{n+1}}{p_n}\cot n\theta \sin\theta\right). \end{aligned} \tag{16}$$

Lemma 7. *Let*

$$x_\alpha = \cos\varphi_\alpha, \qquad \varphi_\alpha = \alpha\pi/n, \qquad \alpha = 0, 1, \ldots, n,$$

$$X_{j,k} = \sum_{\alpha=1}^{n-1}(1-x_\alpha^2)\, U_{j-1}(x_\alpha)\, U_{k-1}(x_\alpha) = \sum_{\alpha=1}^{n-1}\sin j\varphi_\alpha \sin k\varphi_\alpha. \tag{17}$$

Then

$$\begin{aligned} X_{j,k} &= 0 && \text{if } j, k < n, \quad j \neq k, \\ &= n/2 && \text{if } j = k, \quad 0 < j < n, \\ &= 0 && \text{if } j = 0 \quad \text{for all } k \text{ or if } k = 0 \text{ for all } j. \end{aligned} \tag{18}$$

Theorem 7. *If $f_n(x)$ is an approximation to $f(x)$,*

$$f_n(\cos\varphi) = \sum_{k=1}^{n-1} r_k \sin k\varphi \qquad \text{or} \qquad f_n(x) = (1-x^2)^{1/2} \sum_{k=1}^{n-1} r_k U_{k-1}(x)$$

then

$$r_k = \frac{2}{n} \sum_{a=1}^{n-1} f(\cos\varphi_a) \sin k\varphi_a = \frac{2}{n} \sum_{a=1}^{n-1} (1-x_a^2)^{1/2} f(x_a)\, U_{k-1}(x_a) \tag{19}$$

Lemma 8

$$r_i = p_i - \sum_{s=1}^{\infty} p_{2sn-i} + \sum_{s=1}^{\infty} p_{2sn+i} \tag{20}$$

Theorem 8 *If*

$$\delta_n(x) = f(x) - f_n(x)$$

where $f(x)$ and $f_n(x)$ are given by (14) *and* (19), *respectively, then*

$$\begin{aligned} \delta_n(\cos\varphi) &= p_n \sin n\varphi + 2\sin n\varphi \sum_{r=1}^{\infty} p_{n+r} \cos r\varphi \\ \delta_n(\cos\varphi_a) &= 0 \\ \delta_n(\cos\varphi) &\sim p_n \sin n\varphi \left(1 + \frac{2p_{n+1}}{p_n} \cos\varphi\right) \end{aligned} \tag{21}$$

Note that (15) and (20) may be interpreted as the error when p_k in (14) is approximated by the modified trapezoidal rule and the trapezoidal rule, respectively See the comments following (11) Also $\frac{1}{2}(q_k + r_k)$ is an improved approximation for p_k $k = 0, 1, \ldots, n$

For some references pertinent to the material of this section, see von Sanden (1959), Zurmuhl (1964), Elliott (1965), and Cooper (1967)

8 6 Differential and Integral Properties of Expansions in Series of Chebyshev Polynomials of the First Kind

Advantages of expansions in series of the Chebyshev polynomials $T_n(x)$ have been discussed in 8 5 3 Further, in 8 7 we show that the sum of a truncated expansion in series of Chebyshev polynomials can be evaluated in much the same fashion as one computes an ordinary polynomial Now given the coefficients in the expansion for $f(x)$ in series of the shifted polynomials $T_n^*(x)$, one can readily obtain the

coefficients in the corresponding expansions for $[f(x) - f(0)]/x$, $\int_0^x f(t)\,dt$, $\int_0^x t^2 f(t)\,dt$, etc. Such formulas, and others which follow from the material in 8.5.1 and 8.5.2, are developed in 8.6.1. In 8.6.1(10), we prove that the pertinent coefficients may be computed by use of a backward recursion scheme. Similar results for expansions in series of the even and odd Chebyshev polynomials $T_{2n}(x)$ and $T_{2n+1}(x)$, respectively, are taken up in 8.6.2 and 8.6.3, respectively.

We assume throughout that $f(x)$ is restricted to insure that all expansions are valid (see 8.3).

8.6.1. Series of Shifted Chebyshev Polynomials

We suppose that

$$f(x) = \sum_{n=0}^{\infty} b_n T_n^*(x). \tag{1}$$

Throughout this section, as well as in 8.6.2 and 8.6.3, we use the notation

$$\epsilon_0 = 1, \qquad \epsilon_n = 2 \qquad \text{if} \quad n > 0. \tag{2}$$

If $f(0) = 0$, then

$$\begin{aligned} f(x) &= x \sum_{n=0}^{\infty} c_n T_n^*(x), \\ c_n &= 2\epsilon_n \sum_{k=0}^{\infty} (-)^k (k+1)\, b_{k+n+1}, \\ b_0 &= \tfrac{1}{2}c_0 + \tfrac{1}{4}c_1, \qquad b_1 = \tfrac{1}{2}c_0 + \tfrac{1}{2}c_1 + \tfrac{1}{4}c_2, \\ b_n &= \tfrac{1}{4}(c_{n-1} + 2c_n + c_{n+1}), \qquad n = 2, 3, \ldots, \\ \sum_{k=0}^{\infty} (-)^k b_k &= 0. \end{aligned} \tag{3}$$

$$\begin{aligned} xf(x) = \tfrac{1}{4} \Big\{ &(2b_0 + b_1)\, T_0^*(x) + (2b_0 + 2b_1 + b_2)\, T_1^*(x) \\ &+ \sum_{k=2}^{\infty} (b_{k-1} + 2b_k + b_{k+1})\, T_k^*(x) \Big\}. \end{aligned} \tag{4}$$

$$\begin{aligned} x^2 f(x) = \tfrac{1}{16} \Big\{ &(6b_0 + 4b_1 + b_2)\, T_0^*(x) + (8b_0 + 7b_1 + 4b_2 + b_3)\, T_1^*(x) \\ &+ (2b_0 + 4b_1 + 6b_2 + 4b_3 + b_4)\, T_2^*(x) \\ &+ \sum_{k=3}^{\infty} (b_{k-2} + 4b_{k-1} + 6b_k + 4b_{k+1} + b_{k+2})\, T_k^*(x) \Big\}. \end{aligned} \tag{5}$$

$$f(x) = \sum_{n=0}^{\infty} d_n T_n^*(x),$$

$$d_n = 2\epsilon_n \sum_{k=0}^{n} (n + 2k + 1)\, b_{n+2k+1},$$

$$2d_n/\epsilon_n = d_{n+2} + 4(n+1)\, b_{n+1} \tag{6}$$

$$\int_0^x f(t)\,dt = \left\{\frac{b_0}{2} - \frac{b_1}{8} - \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-)^n b_n}{n^2 - 1}\right\} T_0^*(x) + \tfrac{1}{4}(2b_0 - b_2)\, T_1^*(x)$$

$$+ \frac{1}{4}\sum_{n=2}^{\infty} \frac{(b_{n-1} - b_{n+1})}{n}\, T_n^*(x) \tag{7}$$

If $f(0) = 1$,

$$\int_0^x t^{-1}[1 - f(t)]\,dt = \sum_{n=0}^{\infty} d_n T_n^*(x),$$

$$d_0 = -b_1 + \sum_{n=2}^{\infty} (-)^n \left\{2\sum_{k=1}^{n-1} k^{-1} + n^{-1}\right\} b_n,$$

$$d_n = -\frac{b_n}{n} + \frac{2(-)^{n-1}}{n} \sum_{k=n+1}^{\infty} (-)^k b_k, \qquad n = 1, 2, \ldots,$$

$$d_{n+1} = -\frac{n d_n}{n+1} + \frac{b_{n+1} - b_n}{n+1}, \qquad n = 0, 1, \tag{8}$$

$$\int_0^x t f(t)\,dt = \frac{1}{4}\left\{\frac{3b_0}{4} + \frac{b_1}{12} - \frac{19b_2}{48} + 3\sum_{k=3}^{\infty} \frac{(-)^k b_k}{(k^2 - 1)(k^2 - 4)}\right\} T_0^*(x)$$

$$+ \tfrac{1}{16}\{4b_0 + b_1 - 2b_2 - b_3\} T_1^*(x) + \tfrac{1}{32}\{2b_0 + 2b_1 - 2b_3 - b_4\} T_2^*(x)$$

$$+ \frac{1}{16}\sum_{k=3}^{\infty} \left\{\frac{b_{k-2} + 2b_{k-1} - 2b_{k+1} - b_{k+2}}{k}\right\} T_k^*(x) \tag{9}$$

If $f(0) \neq 0$,

$$\int_0^x t^{\mu} f(t)\,dt = x^{\mu+1} \sum_{n=0}^{\infty} e_n T_n^*(x), \qquad R(\mu) > -1,$$

$$\begin{aligned} -\mu e_1 + 2(\mu + 1)e_0 &= 2b_0 - b_1, \\ (n - \mu)e_{n+1} + (n + \mu + 1)e_n &= b_n - b_{n+1}, \qquad n = 1, 2, 3, \ldots, \\ \sum_{n=0}^{\infty} (-)^n b_n &= f(0), \qquad \sum_{n=0}^{\infty} (-)^n e_n = f(0)/(\mu + 1) \end{aligned} \tag{10}$$

This is essentially a generalization of (9). The restriction $f(0) \neq 0$ is not essential. For if $f(0) = 0$, but $f'(0) \neq 0$, then from (3), $f(x) = xh(x)$, $h(x) = \sum_{n=0}^{\infty} c_n T_n^*(x)$. Thus apply (10) with $f(x)$ and μ replaced by $h(x)$ and $\mu + 1$, respectively.

It is of interest to develop a convenient procedure for the evaluation of the e_n's. We first suppose that μ is not a positive integer. Write

$$\begin{aligned} e_n^{(N)} &= -\frac{(n-\mu)}{n+\mu+1} e_{n+1}^{(N)} + \frac{(b_n - b_{n+1})}{n+\mu+1}, \qquad n = 1, 2, \ldots, N, \\ e_{N+k}^{(N)} &= 0, \qquad k = 1, 2, 3, \ldots . \end{aligned} \tag{11}$$

Thus we can evaluate $e_{N-k}^{(N)}$ for $k = 0, 1, \ldots, N-1$ most easily by recursion. The closed form formula

$$e_{N-k}^{(N)} = (-)^k \sum_{r=0}^{\infty} \frac{(-)^r (N-k-\mu)_{k-r}}{(N+\mu-k+1)_{k+1-r}} (b_{N-r} - b_{N-r-1}), \qquad k = 0, 1, \ldots, N-1, \tag{12}$$

is also available. We prove that

$$e_n = \lim_{N\to\infty} e_n^{(N)}, \qquad n = 1, 2, 3, \ldots . \tag{13}$$

For the proof, we first note from the difference equation for e_n given in (10) that

$$\begin{aligned} e_{n+1} &= e_1 H_{n+1} + P_{n+1}, \qquad H_{n+1} = \frac{(-)^n(\mu+2)_n}{(1-\mu)_n}, \\ P_{n+1} &= \frac{(-)^{n+1}(\mu+2)_n}{(\mu+2)(1-\mu)_n} \sum_{m=0}^{n-1} \frac{(-)^m (1-\mu)_m}{(\mu+3)_m} (b_{m+1} - b_{m+2}). \end{aligned} \tag{14}$$

Thus H_{n+1} and P_{n+1} are the homogeneous and particular solutions of this difference equation, respectively. The solution of (11) is of the form

$$e_n^{(N)} = \alpha H_n + P_n \tag{15}$$

where α is a constant. Since $e_{N+1}^{(N)} = 0$, α is determined and

$$e_n^{(N)} = -\frac{P_{N+1}}{H_{N+1}} H_n + P_n .$$

Now

$$q = \lim_{N\to\infty} -\frac{P_{N+1}}{H_{N+1}} = \frac{1}{\mu+2} \sum_{m=0}^{\infty} \frac{(-)^m (b_{m+1} - b_{m+2})(1-\mu)_m}{(\mu+3)_m} .$$

But from (10),

$$b_{m+1} - b_{m+2} = (m + 1 - \mu)e_{m+2} + (m + \mu + 2)e_{m+1},$$

and so

$$q = \frac{1}{\mu + 2}\left[\sum_{m=0}^{\infty} \frac{(-)^m(1-\mu)_m}{(\mu+3)_m}(m + 1 - \mu)e_{m+2}\right.$$

$$\left. + \sum_{m=0}^{\infty} \frac{(-)^m(1-\mu)_m}{(\mu+3)_m}(m + \mu + 2)e_{m+1}\right]$$

$$= -\sum_{m=1}^{\infty} \frac{(-)^m(1-\mu)_m}{(\mu+2)_m} e_{m+1} + \sum_{m=0}^{\infty} \frac{(-)^m(1-\mu)_m}{(\mu+2)_m} e_{m+1} = e_1$$

Thus

$$\lim_{N\to\infty} e_{n+1}^{(N)} = e_1 H_{n+1} + P_{n+1} = e_{n+1}, \qquad n = 0, 1, 2, \ldots,$$

which was to be proved

If μ is a positive integer, say $\mu = s$, the procedure must be slightly altered In this event, we have

$$(n - s)e_{n+1} + (n + s + 1)e_n = b_n - b_{n+1}, \qquad n = 1, 2, \ldots, s, \tag{16}$$

$$ne_{n+s+1} + (n + 1 + 2s)e_{n+s} = b_{n+s} - b_{n+s+1}, \qquad n = 1, 2, 3, \ldots, \tag{17}$$

and from the former,

$$e_n = \frac{(s-n)!\,(s+n)!}{(2s+1)!} \sum_{r=0}^{s-n} \frac{(-)^r(-2s-1)_r}{r!}(b_{s-r} - b_{s-r-1}), \qquad n = 1, 2, \ldots, s \tag{18}$$

Now write

$$e_{n+s}^{(N)} = -\frac{n}{n+1+2s} e_{n+s+1}^{(N)} + \frac{(b_{n+s} - b_{n+s+1})}{n+1+2s}, \qquad n = 1, 2, \ldots, N,$$
$$e_{N+k+s}^{(N)} = 0, \qquad k = 1, 2, 3, \ldots, \tag{19}$$

and so evaluate $e_{N+s}^{(N)}, e_{N+s-1}^{(N)}, \ldots, e_{s+1}^{(N)}$ by recursion We also have

$$e_{N+s-k}^{(N)} = (-)^k \sum_{r=0}^{k} \frac{(-)^r(N-k)_{k-r}}{(N+2s-k+1)_{k+1-r}}(b_{N+s-r} - b_{N+s-r+1}) \tag{20}$$

Then we prove that

$$e_{s+n} = \lim_{N\to\infty} e_{s+N}^{(N)}, \qquad n = 1, 2, 3, \tag{21}$$

The manner of proof is much akin to that for the case when μ is not a positive integer. Using (17), we get

$$e_{n+1+s} = e_{s+1}H_{n+1}^{(s)} + P_{n+1}^{(s)}, \qquad H_{n+1}^{(s)} = \frac{(-)^n(2s+2)_n}{n!},$$

$$P_{n+1}^{(s)} = \frac{(-)^{n+1}(2s+n+1)!}{n!(2s+2)!}\sum_{m=0}^{n-1}\frac{(-)^m m!}{(2s+3)_m}(b_{s+1+m} - b_{s+2+m}).$$

From (19), we have

$$e_{s+n}^{(N)} = \alpha H_n^{(s)} + P_n^{(s)}$$

where α is a constant which is readily determined since $e_{s+N+1}^{(N)} = 0$. Thus,

$$e_{s+N}^{(N)} = -\frac{P_{N+1}^{(s)}}{H_{N+1}^{(s)}}H_n^{(s)} + P_n^{(s)}.$$

With the aid of (16) and (17), we find that

$$\lim_{N\to\infty} -\frac{P_{N+1}^{(s)}}{H_{N+1}^{(s)}} = e_{s+1}$$

which leads to the statement (21).

If

$$\int_0^x f(t)(\ln t)\,dt = \sum_{n=0}^{\infty} a_n T_n^*(x) + (\ln x)\sum_{n=0}^{\infty} g_n T_n^*(x),$$

then

$$g_0 = \frac{b_0}{2} - \frac{b_1}{8} - \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-)^n b_n}{n^2-1}, \qquad g_1 = \tfrac{1}{4}(2b_0 - b_1),$$

$$g_n = (1/4n)(b_{n-1} - b_{n+1}), \qquad n = 2, 3, \ldots,$$

$$a_0 = -\left\{\frac{h_0}{2} - \frac{h_1}{8} - \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-)^n h_n}{n^2-1}\right\}, \tag{22}$$

$$h_n = 2\epsilon_n\sum_{k=0}^{\infty}(-)^k(k+1)g_{k+n+1}, \qquad a_1 = g_1 - 2g_0,$$

$$a_n = -(1/n)\left\{g_n - 2\sum_{k=0}^{\infty}(-)^k g_{k+n+1}\right\}, \qquad n = 2, 3, \ldots.$$

$$\int_0^x \int_0^t f(u)\,du\,dt = \frac{1}{4}\left\{\frac{3b_0}{4} - \frac{b_1}{3} - \frac{3b_2}{16} - \sum_{k=3}^{\infty} \frac{(-)^k b_k}{k^2-4}\right\} T_0^*(x)$$

$$+ \frac{1}{4}\left\{b_0 - \frac{3b_1}{8} - \frac{b_2}{3} + \frac{b_3}{4} - \sum_{k=4}^{\infty} \frac{(-)^k b_k}{k^2-1}\right\} T_1^*(x)$$

$$+ \tfrac{1}{16}\{6b_0 - 4b_2 + b_4\}\, T_2^*(x)$$

$$+ \frac{1}{16}\sum_{k=3}^{\infty} \left\{\frac{(k+1)\,b_{k-2} - 2kb_k + (k-1)\,b_{k+2}}{k(k^2-1)}\right\} T_k^*(x) \tag{23}$$

Let

$$A(x) = \sum_{n=0}^{\infty} a_n T_n^*(x), \qquad B(x) = \sum_{n=0}^{\infty} b_n T_n^*(x), \qquad C(x) = \sum_{n=0}^{\infty} c_n T_n^*(x)$$

If

$$C(x) = A(x)\,B(x),$$

then

$$c_0 = a_0 b_0 + \tfrac{1}{2}\sum_{m=1}^{\infty} a_m b_m$$

$$c_n = a_0 b_n + \sum_{m=1}^{\infty}\left(\frac{b_{n+m}}{2} + \frac{b_{|n-m|}}{\epsilon_{|n-m|}}\right) a_m$$

$$= b_0 a_n + \sum_{m=1}^{\infty}\left(\frac{a_{n+m}}{2} + \frac{a_{|n-m|}}{\epsilon_{|n-m|}}\right) b_m, \qquad n > 0,$$

$$\epsilon_0 = 1, \qquad \epsilon_n = 2 \qquad \text{for} \quad n \geqslant 1 \tag{24}$$

This result is also true if in $A(x)$, $B(x)$, and $C(x)$, $T_n^*(x)$ is replaced by $T_n(x)$ or $T_{2n}(x)$

The above system of equations may be expressed in matrix form as follows Let $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ be infinite matrices where

$$\begin{aligned}
&\alpha_i = a_{i-1} && \alpha_{i1} = \alpha_i \quad \text{for} \quad i = 1, 2, 3, \ldots, \\
&\alpha_{1j} = \tfrac{1}{2}\alpha_j && \text{for} \quad j = 2, 3, \ldots, \\
&\alpha_{ii} = \alpha_1 + \tfrac{1}{2}\alpha_{2i-1} && \text{for} \quad i = 2, 3, \ldots, \\
&\alpha_{ij} = \tfrac{1}{2}(\alpha_{i+j-1} + \alpha_{j-i+1}) && \text{for} \quad j > i \geqslant 2, \\
&\alpha_{ij} = \alpha_{ji}, && i, j, = 2, 3, \ldots,
\end{aligned} \tag{25}$$

and where (25) is also valid if a_i, α_i, and α_{ij} are replaced by b_i, β_i, and β_{ij}, respectively. Let α, β, and γ stand for the infinite vectors

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \end{pmatrix}. \tag{26}$$

Then

$$A\beta = B\alpha = \gamma. \tag{27}$$

8.6.2. Series of Chebyshev Polynomials of Even Order

In this section we suppose that

$$f(x) = \sum_{n=0}^{\infty} b_n T_{2n}(x). \tag{1}$$

Again we make use of the notation 8.6.1(2). Some of the results of this section are essentially restatements of results in 8.6.1 since $T_n^*(x^2) = T_{2n}(x)$. For example, if $f(0) = 0$, then

$$f(x) = x^2 \sum_{n=0}^{\infty} c_n T_{2n}(x), \tag{2}$$

where c_n is given by 8.6.1(3).

$$xf(x) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{2b_n}{\epsilon_n} + b_{n+1} \right) T_{2n+1}(x). \tag{3}$$

$$\begin{aligned} x^2 f(x) &= \tfrac{1}{4}(2b_0 + b_1)\, T_0(x) + \tfrac{1}{4}(2b_0 + 2b_1 + b_2)\, T_2(x) \\ &\quad + \frac{1}{4} \sum_{n=2}^{\infty} (b_{n-1} + 2b_n + b_{n+1})\, T_{2n}(x). \end{aligned} \tag{4}$$

$$\begin{gathered} f'(x) = \sum_{n=0}^{\infty} d_n T_{2n+1}(x), \qquad d_n = 4 \sum_{k=0}^{\infty} (n + k + 1)\, b_{n+k+1}, \\ d_n = d_{n+1} + 4(n + 1)\, b_{n+1}. \end{gathered} \tag{5}$$

$$\int_0^x f(t)\, dt = (b_0 - \tfrac{1}{2}b_1)\, T_1(x) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(b_n - b_{n+1})}{2n + 1}\, T_{2n+1}(x). \tag{6}$$

$$x^{-1}\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} e_n T_{2n}(x),$$

$$e_n = \frac{b_n}{2n+1} + 2(-)^n e_n \sum_{k=n+1}^{\infty} \frac{(-)^k k b_k}{4k^2-1},$$

$$e_{n+1} = -e_n + \frac{b_n - b_{n+1}}{2n+1}, \qquad n = 1, 2, 3, \ldots,$$

$$e_1 = -2e_0 + 2b_0 - b_1 \tag{7}$$

If $f(0) = 1$, then

$$\int_0^x t^{-1}\{1 - f(t)\}\,dt = \sum_{n=0}^{\infty} q_n T_{2n}(x),$$

$$q_0 = \sum_{r=1}^{\infty} (-)^r b_r \left\{\sum_{k=1}^{r-1} \frac{1}{k} + \frac{1}{2r}\right\},$$

$$q_n = -\frac{b_n}{2n} - \frac{(-)^n}{n} \sum_{k=n+1}^{\infty} (-)^k b_k, \qquad n = 1, 2, 3, \ldots,$$

$$q_{n+1} = -\frac{nq_n}{n+1} + \frac{b_{n+1} - b_n}{2(n+1)}, \qquad n = 1, 2, 3, \ldots,$$

$$\sum_{n=0}^{\infty} (-)^n b_n = 1, \qquad \sum_{n=0}^{\infty} (-)^n q_n = 0 \tag{8}$$

$$\int_0^x t f(t)\,dt = \frac{1}{4}\left[b_0 - \frac{b_1}{4} - \sum_{n=2}^{\infty} \frac{(-)^n b_n}{n^2-1}\right] T_0(x) + \tfrac{1}{8}(2b_0 - b_2)\, T_2(x)$$

$$+ \frac{1}{8} \sum_{n=2}^{\infty} \frac{(b_{n-1} - b_{n+1})}{n} T_{2n}(x) \tag{9}$$

$$\int_0^x f(t) \ln t\,dt = \sum_{n=0}^{\infty} a_n T_{2n+1}(x) + \ln x \sum_{n=0}^{\infty} g_n T_{2n+1}(x),$$

$$g_0 = b_0 - \tfrac{1}{2} b_1, \qquad g_n = \frac{1}{2}\left(\frac{b_n - b_{n+1}}{2n+1}\right), \qquad n = 1, 2, \ldots,$$

$$a_n = \frac{g_n}{2n+1} + \frac{2}{2n+1} \sum_{k=1}^{\infty} (-)^k g_{n+k} \tag{10}$$

Let

$$g(x) = \sum_{n=0}^{\infty} b_n T_{2n}(x), \qquad f(x) = \tfrac{1}{2} \sum_{n=0}^{\infty} b_n T_n^*(x),$$

so that

$$\int_0^x t^\nu g(t)\,dt = \int_0^{x^2} t^{(\nu-1)/2} f(t)\,dt, \tag{11}$$

since $T_n^*(x^2) = T_{2n}(x)$. Then evaluation of the integral on the left follows from 8.6.1(10) if there we replace b_n, μ, and x by $\frac{1}{2}b_n$, $\frac{1}{2}(\nu - 1)$, and x^2, respectively.

$$\int_0^x \int_0^t f(u)\,du\,dt = \frac{1}{4}\left[b_0 - \frac{3b_1}{4} + \sum_{n=2}^{\infty} \frac{(-)^n b_n}{n^2 - 1}\right] T_0(x) + \tfrac{1}{24}[6b_0 - 4b_1 + b_2] T_2(x)$$

$$+ \frac{1}{8} \sum_{n=2}^{\infty} \frac{[(2n+1)\,b_{n-1} - 4nb_n + (2n-1)\,b_{n+1}]}{n(4n^2-1)} T_{2n}(x). \tag{12}$$

$$\int_0^x \int_0^t uf(u)\,du\,dt = \frac{1}{4}\left[\frac{b_0}{2} - \frac{b_1}{4} - \frac{b_2}{12} - \sum_{n=3}^{\infty} \frac{(-)^n b_n}{n^2 - 1}\right] T_1(x)$$

$$+ \tfrac{1}{96}[4b_0 - b_1 - 2b_2 + b_3]\, T_3(x)$$

$$+ \frac{1}{16} \sum_{n=2}^{\infty} \frac{[(n+1)\{b_{n-1} - b_{n+1}\} - n\{b_n - b_{n+2}\}]}{n(n+1)(2n+1)} T_{2n+1}(x). \tag{13}$$

8.6.3. Series of Chebyshev Polynomials of Odd Order

We suppose throughout that

$$f(x) = \sum_{n=0}^{\infty} b_n T_{2n+1}(x). \tag{1}$$

The notation of 8.6.1(2) is again used. If

$$f(x) = x \sum_{n=0}^{\infty} c_n T_{2n}(x),$$

then

$$c_n/\epsilon_n + \tfrac{1}{2}c_{n+1} = b_n, \qquad c_n = \epsilon_n \sum_{k=0}^{\infty} (-)^k b_{n+k},$$

$$\sum_{n=0}^{\infty} (-)^n (2n+1) b_n = \sum_{n=0}^{\infty} (-)^n c_n. \tag{2}$$

$$xf(x) = \tfrac{1}{2}b_0 T_0(x) + \tfrac{1}{2} \sum_{n=1}^{\infty} (b_{n-1} + b_n)\, T_{2n}(x). \tag{3}$$

$$x^2 f(x) = \tfrac{1}{4}(3b_0 + b_1)\, T_1(x) + \tfrac{1}{4} \sum_{n=1}^{\infty} (b_{n-1} + 2b_n + b_{n+1})\, T_{2n+1}(x) \tag{4}$$

$$f(x) = \sum_{n=0}^{\infty} d_n T_{2n}(x), \qquad d_n = \epsilon_n \sum_{k=0}^{\infty} (2n + 2k + 1)\, b_{n+k},$$

$$d_{n+1} = \frac{2d_n}{\epsilon_n} + 2(2n + 1)\, b_n \tag{5}$$

$$\int_0^x f(t)\, dt = \frac{1}{4}\left\{b_0 + \sum_{k=1}^{\infty} \frac{(-)^k(2k + 1)\, b_k}{k(k + 1)}\right\} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{(b_{n-1} - b_n)}{n}\, T_{2n}(x) \tag{6}$$

$$\int_0^x t^{-1} f(t)\, dt = \sum_{n=0}^{\infty} r_n T_{2n+1}(x),$$

$$r_n = \frac{b_n}{2n + 1} + \frac{2}{2n + 1} \sum_{k=1}^{\infty} (-)^k b_{n+k},$$

$$\sum_{n=0}^{\infty} (-)^n (2n + 1)\, b_n = \sum_{n=0}^{\infty} (-)^n (2n + 1)\, r_n \tag{7}$$

$$\int_0^x t f(t)\, dt = \frac{(b_0 - b_1)}{4}\, T_1(x) + \frac{1}{4} \sum_{n=1}^{\infty} \frac{(b_{n-1} - b_{n+1})}{2n + 1}\, T_{2n+1}(x) \tag{8}$$

Let

$$g(x) = \sum_{n=0}^{\infty} d_n T_{2n+1}(x) = x \sum_{n=0}^{\infty} c_n T_{2n}(x) \qquad f(x) = \tfrac{1}{2} \sum_{n=0}^{\infty} c_n T_n^*(x),$$

where the c_n's are found from (2) with b_n replaced by d_n Then

$$\int_0^x t^{\nu} g(t)\, dt = \int_0^{x^2} t^{\nu - 2} f(t)\, dt \tag{9}$$

and evaluation of the integral on the left follows from 8 6 1(10) if there we replace b_n, μ, and x by $\frac{1}{2}c_n$, $\frac{1}{2}\nu$, and x^2, respectively

Let

$$A(x^2) = \int_0^x f(t) \ln t\, dt = \int_0^x t \ln t \sum_{n=0}^{\infty} c_n T_{2n}(t)\, dt$$

where c_n is defined in (2) Replace t by $\tau^{1/2}$ and x by $y^{1/2}$ So

$$A(y) = \frac{1}{4} \int_0^y \ln \tau \sum_{n=0}^{\infty} c_n T_n^*(\tau)\, d\tau$$

which can be evaluated with the aid of 8.6.1(22). Indeed, if we now write

$$f(x) = \sum_{n=0}^{\infty} B_n T_{2n+1}(x),$$

$$A(x) = \tfrac{1}{4} \sum_{n=0}^{\infty} a_n T_{2n}(x) + \tfrac{1}{2}(\ln x) \sum_{n=0}^{\infty} g_n T_{2n}(x),$$

then

$$g_0 = \frac{c_0}{2} - \frac{c_1}{8} - \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-)^n c_n}{n^2 - 1}, \qquad g_1 = B_0 + 2 \sum_{k=1}^{\infty} (-)^k B_k,$$

$$g_n = \frac{1}{2n}(B_{n-1} - B_n), \qquad n = 2, 3, \ldots,$$

$$a_0 = -\left\{\frac{h_0}{2} - \frac{h_1}{8} - \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-)^n h_n}{n^2 - 1}\right\}, \tag{10}$$

$$h_n = 2\epsilon_n \sum_{k=0}^{\infty} (-)^k (k+1) g_{k+n+1},$$

$$a_1 = g_1 - 2g_0, \qquad a_n = -(1/n)\left\{g_n - 2 \sum_{k=0}^{\infty} (-)^k g_{k+n+1}\right\}, \qquad n = 2, 3, \ldots,$$

where c_n is defined in (2) with b_n replaced by B_n.

$$\int_0^x \int_0^t f(u)\, du\, dt = \frac{1}{4}\left[\frac{b_0}{2} - b_1 + \sum_{n=2}^{\infty} \frac{(-)^n (2n+1) b_n}{n(n+1)}\right] T_1(x)$$

$$+ \frac{1}{8} \sum_{n=1}^{\infty} \frac{[(n+1)\, b_{n-1} - (2n+1)\, b_n + n b_{n+1}]}{n(n+1)(2n+1)} T_{2n+1}(x). \tag{11}$$

8.7. A Nesting Procedure for the Computation of Expansions in Series of Functions Where the Functions Satisfy Linear Finite Difference Equations

Suppose we have to evaluate

$$f_N(x) = \sum_{n=0}^{N} a_n x^n, \tag{1}$$

an approximation to

$$f(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{2}$$

Here our functions are the monomials x^n Define the backward recurrence scheme

$$B_n = a_n + xB_{n+1}, \qquad n = N, N-1, \quad , 1, 0, \qquad B_{N+1} = 0 \tag{3}$$

We then get the well known result

$$f_N(x) = B_0 \tag{4}$$

It is of interest to examine the effect of a round-off error in a_n when evaluating $f_N(x)$ in this manner We note that the general solution of

$$u_n = a_n + xu_{n+1} \tag{5}$$

is

$$u_n = \alpha x^{-n} - x^{-n} \sum_{k=0}^{n-1} a_k x^k \tag{6}$$

where α is a constant Thus a rounding error δ_n in a_n or B_n produces an error $\epsilon_s(n)$ in B_s for $s \leqslant n$ given by

$$\epsilon_s(n) = \alpha x^{-s} - x^{-s} \sum_{k=0}^{s-1} \delta_k x^k \tag{7}$$

Also

$$0 = \alpha x^{-n-1} - x^{-n-1} \sum_{k=0}^{n-1} \delta_k x^k \tag{8}$$

We can solve the latter for α and so find

$$\epsilon_s(n) = x^{-s} \sum_{k=s}^{n} \delta_k x^k \tag{9}$$

The error in $f(x)$ is (9) with $s = 0$, which is exactly that produced by an error δ_k in a_k when the series is summed in the usual fashion Thus, if B_n is rounded to the same number of places as a_n, the maximum rounding error in $f_N(x)$ is only doubled If one or two guard units are retained in a_n or B_n, this error may be made negligible when compared with the truncation error due to the approximation of $f(x)$ by $f_N(x)$ The above analysis shows that although the error in B_n may become quite large when N is large, the error in $f_N(x)$ is the same as that obtained if we form the required powers of x and sum according to (1) The computation of $f_N(x)$ by the backward recurrence scheme requires only N additions and N multiplications An alternative method for the computation of $f_N(x)$ is to use (1) where the needed powers of x are obtained

by recurrence from $x^n = x \cdot x^{n-1}$. This approach requires $(2N - 1)$ multiplications and N additions and so is not as economical as the backward recursion scheme.

To generalize the preceding results, let

$$f_N(x) = \sum_{k=0}^{N} a_k p_k(x),$$

$$p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x) = 0, \qquad n \geqslant 1, \tag{10}$$

where α_n and β_n may depend on both n and x. Here $p_k(x)$ need not be a polynomial. Consider the backward recursion system

$$\begin{aligned} B_n &= -\alpha_n B_{n+1} - \beta_{n+1} B_{n+2} + a_n, \qquad n = N, N-1, \ldots, 1, 0, \\ B_{N+1} &= B_{N+2} = 0. \end{aligned} \tag{11}$$

Then

$$f_N(x) = B_0 p_0(x) + B_1\{p_1(x) + \alpha_0 p_0(x)\}. \tag{12}$$

This is easily proved by solving for a_n from (11), substituting in $f_N(x)$ and using the recurrence formula in (10). The extension of this principle to the case where the sequence $\{p_k(x)\}$ satisfies a higher order difference equation is direct. Suppose

$$p_{n+1}(x) + \alpha_{n,0} p_n(x) + \alpha_{n,1} p_{n-1}(x) + \cdots + \alpha_{n,r} p_{n-r}(x) = 0, \qquad r < N. \tag{13}$$

Consider

$$\begin{aligned} B_n &= -\alpha_{n,0} B_{n+1} - \alpha_{n+1,1} B_{n+2} - \alpha_{n+2,2} B_{n+3} - \cdots - \alpha_{n+r,r} B_{n+r+1} + a_n, \\ B_{N+1} &= B_{N+2} = \cdots = 0. \end{aligned} \tag{14}$$

Then

$$f_N(x) = \sum_{s=0}^{r} B_s\{p_s + \alpha_{s-1,0} p_{s-1} + \cdots + \alpha_{s-1,s-1} p_0\}. \tag{15}$$

We now show that a similar nesting procedure is available for evaluation of $f_N'(x)$. It is sufficient to treat the system (10), (11). We have

$$f_N'(x) = (B_0 + B_1\alpha_0)\, p_0'(x) + B_1 p_1'(x) + B_1\alpha_0' p_0(x) - g_N(x),$$

$$g_N(x) = \sum_{k=0}^{N-1} b_k p_k(x), \qquad b_k = B_{k+1}\alpha_k' + B_{k+2}\beta_{k+1}', \tag{16}$$

and $g_N(x)$ is readily evaluated using (10)–(12) Thus, let

$$C_n = -\alpha_n C_{n+1} - \beta_{n+1} C_{n+2} + b_n, \qquad n = N-1, N-2, \ldots, 1, 0,$$
$$C_N = C_{N+1} = 0 \tag{17}$$

Then

$$f_N(x) = (B_0 + B_1\alpha_0)\, p_0(x) + B_1 p_1(x) + B_1\alpha_0 p_0(x)$$
$$- C_0 p_0(x) - C_1\{p_1(x) + \alpha_0 p_0(x)\} \tag{18}$$

The algorithm (10)–(12) for the evaluation of $f_N(x)$ when $p_n(x)$ is the Chebyshev polynomial $T_n(x)$ is due to Clenshaw (1955) Its importance lies in the fact that series of Chebyshev polynomials can be evaluated in a manner very much like that a polynomial Thus the conversion of a series of Chebyshev polynomials to an ordinary polynomial is not necessary for its evaluation The extension of the Clenshaw procedure to the calculation of $f_N(x)$ when $p_n(x)$ is a polynomial and has the recurrence formula (10) is due to Smith (1965) We should like to emphasize that the treatment given here is quite general since $p_n(x)$ need not be a polynomial Thus, for example, the algorithm (10)–(12) is applicable to sum an expansion in series of Bessel functions

When $p_k(x)$ satisfies a three-term recurrence formula as in (10), we study the growth of round-off errors in the evaluation of $f_N(x)$ by (12) after the manner of Smith (1965) Let ϵ_n be the error introduced during the calculation of B_n from B_{n+1} and B_{n+2} due to rounding and inaccuracies in the coefficients α_n, β_{n+1}, and a_n If the total error in B_n is E_n, then

$$B_n + E_n = -\alpha_n(B_{n+1} + E_{n+1}) - \beta_{n+1}(B_{n+2} + E_{n+2}) + a_n + \epsilon_n, \tag{19}$$

and in view of (11),

$$E_n = -\alpha_n E_{n+1} - \beta_{n+1} E_{n+2} + \epsilon_n \tag{20}$$

Thus E_n and B_n satisfy the same recurrence system if in (11) we replace a_n by ϵ_n Since $E_n = 0$ for $n > N$, we have

$$E_0 p_0(x) + E_1\{p_1(x) + \alpha_0 p_0(x)\} = \sum_{k=0}^{n} \epsilon_k p_k(x) \tag{21}$$

So an error introduced at the nth step only contributes an error $\epsilon_n p_n(x)$ to the final answer and does not increase the errors introduced in the subsequent n steps Notice that the total error in the calculation of $f_N(x)$ by (12) is the same as that obtained by evaluating $f_N(x)$ as the

sum in (10) when we interpret ϵ_n as the error in a_n. Similar remarks pertain to the evaluation of $f'_N(x)$ by the algorithms (16)–(18). An error analysis for the evaluation of $f_N(x)$ by (10)–(12) when $p_n(x) = T_n(x)$ has been given by Clenshaw (1955). See Cooper (1967) also.

For the case of expansions in series of Chebyshev polynomials of the first kind, the algorithms (10)–(12) and (16)–(18) can be summarized as follows:

$$f_N(x) = \sum_{k=0}^{N} a_k p_k(x), \qquad f'_N(x) = \sum_{k=1}^{N} a_k p'_k(x),$$

$$p_{n+1}(x) + \alpha_n p_n(x) + p_{n-1}(x) = 0. \tag{22}$$

$$\begin{gathered} B_n = -\alpha_n B_{n+1} - B_{n+2} + a_n, \qquad n = N, N-1, \ldots, 0, \\ B_{N+1} = B_{N+2} = 0, \\ C_n = -\alpha_n C_{n+1} - C_{n+2} + b_n, \qquad n = N-1, N-2, \ldots, 0, \\ C_N = C_{N+1} = 0. \end{gathered} \tag{23}$$

$p_n(x)$	α_n	b_n	$f_N(x)$	$f'_N(x)$
$T_n(x)$	$-2x$	$-2B_{n+1}$	$B_0 - xB_1$	$-B_1 - C_0 + xC_1$
$T_{2n}(x)$	$-2(2x^2 - 1)$	$-8xB_{n+1}$	$B_0 - (2x^2 - 1)B_1$	$-4xB_1 - C_0 + (2x^2 - 1)C_1$
$T_{2n+1}(x)$	$-2(2x^2 - 1)$	$-8xB_{n+1}$	$x(B_0 - B_1)$	$B_0 - B_1 - x(C_0 - C_1)$
$T_n^*(x)$	$-2(2x - 1)$	$-4B_{n+1}$	$B_0 - (2x - 1)B_1$	$-2B_1 - C_0 + (2x - 1)C_1$

(24)

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NOTATION INDEX

The system of notation employed to locate specific material lists, in the following order, chapter, section, and subsection (if any).

EXAMPLE. 2.10.3 means Chapter II, tenth section, third subsection. A number in parentheses refers to an equation.

EXAMPLE. 2.10.3(4) refers to the fourth equation in 2.10.3. An equation number in parentheses, standing by itself, refers to the equation of the particular section or subsection in which the reference occurs.

EXAMPLE. A reference to (4) in 2.10.3 means the fourth equation of 2.10.3.

There is much ad hoc notation which is explained in the text near where it occurs. Data of this kind are excluded in the index.

In the listings below, the numbers refer to the equations which define the functions in accordance with the discussion given above.

A

B

C

D

E

Greek Letters

β a parameter used in connection with ${}_pF_q$ to denote $q + 1 - p$ 5 11 1(5)

$\Gamma^{\lambda\,p}_{q+1}(k)$ 5 11 1(14)

$\Gamma(a\ z)$ complementary incomplete gamma function 6 2 11(2)

$\Gamma(z)$ gamma function 2 1(1)

γ Euler–Mascheroni constant 2 1(14) See also Chapter XVII Table 1 In the literature this is often referred to as simply Euler s constant

$\gamma(a\ z)$ incomplete gamma function 6 2.11(1)

$\Delta^{m\,n}_q(t)$ $\Delta(t)$ 5 9 2(1)

$\delta = zD = zd\ dz$ a derivative operator 2 9(1)

δ_{mn} Kronecker delta function 8 1(6)

ϵ a parameter used in connection with $G^{m\,n}_{p\,q}(z)$ 5 7(2)

ϵ_n a parameter such that $\epsilon_n = 1$ if $n = 0$ $\epsilon_n = 2$ if $n > 0$

$\zeta(z)$ Riemann zeta function 2 10 1(7)

η a parameter used in connection with $G^{m\,n}_{p\,q}(z)$ to denote $\frac{3}{2}q - \frac{1}{2}p - m - n + 2$ 5 7(1)

$\theta(l\ r) = \theta^{m\,n}_p(l\ r)$ $\bar\theta(l\ r) = \bar\theta^{m\,n}_p(l\ r)$ 5 9 2(14)

λ a parameter used in connection with Jacobi and extended Jacobi polynomials to denote $\alpha + \beta + 1$ 8 2(1)

ν a parameter used in connection with $G^{m\,n}_{p\,q}(z)$ to denote $q - m - n$ 5 7(1)

ρ a parameter used in connection with $G^{m,n}_{p,q}(z)$ to denote $m + n - \frac{1}{2}(p + q)$ 5 7(1)

σ a parameter used in connection with $G^{m\,n}_{p\,q}(z)$ to denote $q - p$ 5 7(1)

τ a parameter used in connection with $G^{m\,n}_{p\,q}(z)$ to denote $m + n - p$ 5 7(1)

$\Phi(a\ c\ z)$ confluent hypergeometric function 4 4(4)

$\Phi(h\ \lambda) = \Phi^{m\,n}_{p\,q}(h\ \lambda)$ $\bar\Phi(h\ \lambda) = \bar\Phi^{m\,n}_{p\,q}(h\ \lambda)$ 5 9 2(15)

$\psi(a\ c\ z)$ confluent hypergeometric function 4 2(7)

$\psi(h\ \lambda) = \psi^{m\,n}_{p\,q}(h\ \lambda)$ $\bar\psi(h\ \lambda) = \bar\psi^{m\,n}_{p\,q}(h\ \lambda)$ 5 9 2(16)

$\psi(z)$ logarithmic derivative of the gamma function 2.4(1)

$\Omega(k) = \Omega^{m\,n}_q(k)$ $\bar\Omega(k) = \bar\Omega^{m\,n}_q(k)$ 5 9 2(5)

Miscellaneous Notations

$z = x + iy$ $i = (-1)^{1/2}$ is a complex number x and y real

$\bar z = x - iy$

$R(z) = x =$ real part of z

$I(z) = y =$ imaginary part of z

$|z| =$ absolute value of z $|z| = (x^2 + y^2)^{1/2}$

$\arg z =$ argument of z $\tan(\arg z) = y/x$

$\ln z =$ principal value of the natural logarithm of z

$\ln z = \ln|z| + i \arg z$ $-\pi < \arg z \leq \pi$

$z^a = e^{a \ln z}$ with $\ln z$ defined as above

$(a)_k = \Gamma(a + k)/\Gamma(a)$ see 2 1(7)

$\binom{m}{n} =$ binomial coefficient $= m!/[n!(m - n)!]$ 2.1(11)

m as in $A^{(m)}(z)$ usually means $d^mA(z)/dz^m$ e g $\psi^{(m)}(z) = d^m\psi(z)/dz^m$

$[x] =$ largest integer contained in x $x > 0$

$(-)^n = (-1)^n$ n an integer or zero

$P\,\mathrm{v}\int$ means Cauchy principal value of an integral

$\sim$ means asymptotic equality see 1 2(6) This symbol is sometimes used to denote approximate equality e g $\pi \sim 3\ 14$

$\alpha_p = \prod_{j=1}^{p} \alpha_j$ and $(\alpha_p)_k = \prod_{j=1}^{p} (\alpha_j)_k$ are frequently used in connection with generalized hypergeometric functions and G-functions

$*$ as in ${}_pF_q\left(\begin{matrix}\alpha_p \\ 1 + \rho_q - \rho_h^*\end{matrix}\middle|\, z\right)$ $h = 0\ 1\ \ldots\ q$ means to omit the parameter $1 + \rho_j - \rho_h$ when $j = h$ Also $(\alpha_p - \alpha_h)^*$ stands for $\prod_{j=1,\, j\neq h}^{p} (\alpha_j - \alpha_h)$ etc

Miscellaneous Notations (cont.)

d as in *6d* means six decimals

s as in *7s* means seven significant figures

The notation $x = 0(0.2)\ 2.0$, for example, is used in connection with the description of tabular data and means that data are given for x from 0 to 2.0 in gaps of 0.2

A number in parentheses following a base numerical number indicates the power of ten by which the base number is to be multiplied. In illustration, 2.35786(−3) means $2.35786 \cdot 10^{-3}$

SUBJECT INDEX

The following index is for Volume I only. Volume II has a cumulative Subject Index for both volumes.

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